SPATIAL OBJECT MODELING

IN FUZZY TOPOLOGICAL SPACES

with Applications to Land Cover Change

Xinming Tang
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DISSERTATION

to obtain
the doctor's degree at the University of Twente,
on the authority of the rector magnificus,
Prof.dr. F.A. van Vught,
on account of the decision of the graduation committee,
to be publicly defended
on Friday 9 January 2004, at 15.00 hrs

by

Xinming Tang

Born on 5 December 1966
in Jiangsu, China

This thesis is approved by
Prof. Dr. W. Kainz, promotor

To my wife and daughter
and
to my father

Abstract

Currently most GISs represent natural phenomena by crisp spatial objects. In fact many natural phenomena have fuzzy characteristics. The representation of these objects in the crisp form greatly simplifies the processing of spatial data. However, this simplification cannot describe these natural phenomena precisely, and it will lead to loss of information in these objects. In order to describe natural phenomena more precisely, the fuzziness in these natural phenomena should be considered and represented in a GIS. This will allow the derivation of better results and a better understanding of the real world to be achieved.

The central topic of this thesis focuses on the accommodation of fuzzy spatial objects in a GIS. Several issues are discussed theoretically and practically, including the definition of fuzzy spatial objects, the topological relations between them, the modeling of fuzzy spatial objects, the generation of fuzzy spatial objects and the utilization of fuzzy spatial objects for land cover changes.

A formal definition of crisp spatial objects has been derived based on the highly abstract mathematics such as set theory and topology. Fuzzy set theory and fuzzy topology are the ideal tools for defining fuzzy spatial objects theoretically, since fuzzy set theory is a natural extension of classical set theory and fuzzy topology is built based on fuzzy sets. However, owing to the extension, several properties holding between crisp sets do not hold for fuzzy sets.

The key issue of a fuzzy spatial object is its boundary. Three definitions of fuzzy boundary are revisited and one is selected for the definition of fuzzy spatial objects. Besides the fuzzy boundary, several notions such as the core, the internal, the fringe, the frontier, the internal fringe and the outer of a fuzzy set are defined in fuzzy topological space. The relationships between these notions and the interior, the boundary and the exterior of a fuzzy set are revealed. In general, the core is the crisp subset of the interior, and the fringe is a kind of boundary but shows a finer structure than the boundary of a fuzzy set in fuzzy topological space. These notions are all proven to be topological properties of a fuzzy topological space.

The definition of a simple fuzzy region is derived based on the above topological properties. It is discussed twice in the thesis. Firstly, the definition of a simple fuzzy region is given in a special fuzzy topological space called crisp fuzzy topological space, since most topological properties of a fuzzy set in the fuzzy topological space are the same as those in crisp topological space. A formal definition of a simple fuzzy region is proposed based on the discussion of the topological properties, besides the interior, the boundary and the exterior, of a fuzzy set in the general fuzzy topological space. A crisp simple region is a special form of a simple fuzzy region.

One of the fundamental properties between fuzzy spatial objects is the topological relations. This topic is intensively discussed in the thesis. The problem of the

9-intersection approach for identifying topological relations between fuzzy spatial objects is revealed. In order to derive the topological relations between fuzzy spatial objects, the 9-intersection approach is updated into the 3*3-intersection approach in the crisp fuzzy topological space. Furthermore, the 4*4-intersection matrix is built up by using the topological properties of fuzzy sets, and the 5*5-intersection matrix can be built up based on a certain condition in crisp fuzzy topological space. These matrices are then updated in the general fuzzy topological space, based on topological properties, other than the interior, the boundary and the exterior, of two fuzzy sets. Two 3*3-intersection and one 4*4-intersection matrices are introduced in the general fuzzy topological space. The topological relations between simple fuzzy regions can be identified based on the topological invariants in the intersections of the matrices. Using the empty/non-empty topological invariants in the intersections, 44 and 152 relations are derived between two simple fuzzy regions.

The modeling of fuzzy spatial objects should be done not only for simple fuzzy regions, but also for fuzzy lines and fuzzy points. In order to model fuzzy lines and fuzzy points and the topological relations between fuzzy spatial objects, a fuzzy cell is proposed and a fuzzy cell complex can be constructed from fuzzy cells. A fuzzy region, a fuzzy line and a fuzzy point are then defined according to this structure. The relations between these fuzzy spatial objects are identified. The fuzzy cell complex structure constitutes a theoretic framework, since it can easily model the fuzzy spatial objects.

After proposing the theoretic framework for fuzzy spatial object modeling, the thesis addresses several practical issues on applying fuzzy spatial objects. The first issue is how to generate fuzzy spatial objects. A composite method is proposed for the generation of fuzzy land cover objects. It involves several steps, from designing membership functions to classification and refining the membership values of fuzzy land cover objects.

Another practical issue is how to retrieve fuzzy spatial objects, particularly on the basis of topological relations. In traditional GIS, the query operators are defined based on the relatively small number of topological relations. However, there are many topological relations between fuzzy spatial objects. In order to query fuzzy spatial objects, the query operators are proposed and formalized based on the common-sense operators in traditional GIS. The 44 or 152 topological relations are grouped into these operators by four different methods. These methods constitute a relatively complete covering for querying fuzzy spatial objects so as to meet the different application requirements.

The third practical issue is how to use fuzzy spatial objects in real applications. Since the dynamics of land covers is a very important topic in China, the focus lies on calculating changes of land covers. Sanya city, located in south China, is selected as the test area. A fuzzy reasoning method is proposed for calculating land cover changes. It shows that, with fuzzy representation, not only can a better result be achieved for the land cover changes, but also the details of changes can be revealed.

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Chapter One

Introduction

1.1 Background

1.1.1 Uncertainty and related theories

In the Merriam-Webster Dictionary, uncertainty is explained as something uncertain: indefinite, indeterminate; not certain to occur; not reliable; not known beyond doubt; not having certain knowledge; not clearly identified or defined; not constant. Almost all the information that we possess about the real world is uncertain, incomplete and imprecise. From the general point of view, uncertainty may include the following aspects (Worboys 1998):

- Inaccuracy and error: deviations from true values;
- Vagueness: imprecision in concepts used to describe the information;
- Incompleteness: lack of relevant information;
- Inconsistency: conflicts arising from the information;
- Imprecision: limitation on the granularity or resolution at which the observation is made or the information is represented.

Error, as one aspect of the uncertainties, represents bias from true values. The error is 1% if 99 out of 100 events are the true value, which is a singleton value normally. Vagueness can be the inherent nature of an object, or result from imprecise knowledge or from the methods of observation. Incompleteness is caused by lack of relevant information, for example, lack of sufficient information to determine the location of the center of a city. Inconsistency expresses paradox in some events. Imprecision usually arises because of limitations on the granularity or resolution at which the observation is made. For example, a Digital Elevation Model (DEM) with a resolution of 100 m will mean the loss of detail within 100 m.

Error has been tackled using probability theory ever since the 17th century. Bayesian theory is a classical model for handling errors in events. In order to deal with vagueness, Zadeh proposed the famous fuzzy set theory in 1965. The fuzzy set and fuzzy logic are the most powerful tool for solving these fuzzy problems. Since then, many theories have been proposed for solving different aspects of the uncertainty problems by addressing different facets. Possibility theory, introduced by Zadeh (1978), in connection with fuzzy set theory, allows a reasoning to be carried out on imprecise or vague knowledge, making it possible to deal with uncertainties in this knowledge. The Dempster-Shafer evidence theory (Shafer 1976) is like the Bayesian probability theory. It relies on degrees of belief to represent imprecision in events. Unlike the Bayesian theory, however, it permits us to assign degrees of plausibility to subsets of events. In Bayesian theory, we construct a probability distribution over all individual singleton events, but in evidence theory a distribution is constructed over all subsets of events. Rough set theory, introduced by Pawlak (1991), represents the uncertainty of an event by the approximation of sets using a collection of sets. It is widely adopted in the field of data mining since it is powerful for reasoning based on incomplete information. Worboys (1998) also infers about the integration of imprecision in data in terms of spatial resolution.

1.1.2 Fuzzy sets and fuzzy spatial objects

While different theories are proposed for solving different problems on uncertainties, fuzzy set theory is emphasized for representing spatial objects. The idea of fuzzy set is to express the facts in human knowledge, such as:

(1) Partial membership to a class (such as “almost true”);
(2) Categories with poorly defined boundaries (“young” or “far”);
(3) Gradual change from one situation to another (transition from “warm” to “hot” as the temperature changes);
(4) Use of approximate values (“about 12 years”).

Zadeh (1965) generalized a fuzzy set from classical set theory by allowing intermediate situations between the whole and nothing. For a fuzzy (sub)set, a membership function is defined to describe the degree of membership of an element to a class. The membership value ranges from 0 to 1, where 0 shows that the element does not belong to a class, 1 means “belong”, and other values indicate the degree of membership to a class. The difference between fuzzy set and crisp set lies in the concept that the membership function has replaced the characteristic function of a set. A fuzzy set can represent the elements in a class with a degree of membership to that class. Fuzzy set theory has been built as a natural extension of classic set theory. It provides a means of representing and handling the vagueness of an object and imperfectly described knowledge.

When we investigate and analyze natural phenomena, we always describe them by some terminologies of human knowledge. Many terminologies express a general characteristic of an object, *i.e.*, they possess a definite connotation and cover a large extent of certain phenomena, such as “young” and “old”, “large” and “small”. Many notions of spatial features fall into this category, for instance, urban and rural, physical geographic region, forest and grassland. The phenomena corresponding to these notions are distributed continuously in space and have a characteristic in common – they have indeterminate boundaries. In the discussion of spatial objects (Freksa and Barkowsky 1996, Clementini and Di Felice 1996, Hadzilacos 1996, Lagacherie *et al.* 1996), fuzzy spatial objects are those with indeterminate boundaries. The indeterminate boundary of a spatial object refers to the fact that there is some degree of membership of points belonging to that spatial object. According to the idea and explanation of fuzzy sets, fuzzy set theory is an ideal tool for handling these natural phenomena because of its capability to represent the indeterminate boundaries of these objects.

In GIS, a spatial object is usually subdivided into three parts: spatial, non-spatial (mainly referred to as attributes) and temporal. Fuzziness may exist in all of these aspects. We can distinguish the following fuzziness of spatial objects: fuzziness in object class, fuzziness in object attributes, fuzziness in location and fuzziness in time.

The fuzziness in object class can be interpreted as a category problem. It is usually caused by ambiguous definitions. For example, grassland can be defined as “an area most of which is covered by grass”, in which the term “most” is not clear. The vagueness existing in spatial objects is the key factor that raises ambiguous definitions. Attribute fuzziness can be regarded as a category fuzziness if taking attributes as attribute classes. Location fuzziness rises (1) we know the precise locations of the geographic objects, including the possibly gradual transitions between them, but we are uncertain how to classify them. This fuzziness can be regarded as class fuzziness. Location fuzziness can also be because of (2) spatially imprecise definitions. Coarse resolution will cause the imprecision of information representation. Even if we can define category classes clearly, it is impossible to classify them crisply since they are imprecisely represented. Temporal fuzziness may be incomplete temporal information, such as not knowing exactly when something happens.

1.1.3 Fuzziness in land cover

Land use and land cover (LULC), most of which is obtained from the classification results of satellite images or air-photos, may be a good example of a fuzzy spatial object. Since many researchers consider that image classification usually derives land cover and that land use denotes the real use of land (for example, grassland and buildings classified from satellite images can be regarded as land cover but the real land use is garden), the term “land cover” will be adopted in this thesis. After classification each pixel in the image is assigned to a particular land cover type. Any pixel belongs to one and only one type and the whole area of that pixel is assigned to that type. In short, the decision is a Boolean assignment of each pixel to a class (Fisher 1996). The normal procedure of a classification is to develop a set of training areas that represent each of the land cover types, and then to use statistical methods from those areas as a base for some numerical procedure to attempt to assign each pixel to a type. A number of methods can be adopted, but some variants of the maximum likelihood method are perhaps the most widely used classifiers. The methods determine the probability of a pixel belonging to all classes and include a decision rule that saves only the name of the most likely class for the pixel. One variant includes a chi-square test to determine the confidence with which a pixel can be classified, and this can be used to leave those pixels unclassified where classification is in doubt.

However, this is always an approximation of reality. In fact, the landscape is not made up of little rectangular plots of uniform land use that suddenly change their size to match resolutions such as 10m, 20m or 30m. At some scales, all pixels actually contain a number of different contributing land use types (Figure 1.1). The significance of the contribution is clearly dependent on the sensor resolution. It would therefore be more correct to say that a pixel has some levels of possibility of belonging to certain land cover types.

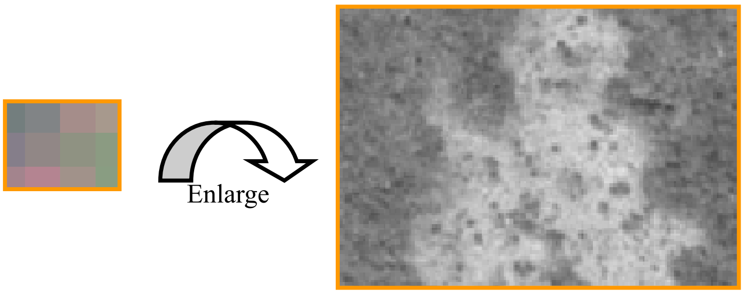


Figure 1.1 Fuzziness in pixels

Furthermore, in general land cover is continuously distributed in nature, and there is seldom a clear boundary between different land covers. For instance, it quite often happens that there are no clear boundaries between shrub and grassland (Figure 1.2). In other words, the artificial crisp division between these land covers is less accurate than the indeterminate boundary in terms of representing of land cover objects. Coarse resolutions just reinforce this characteristic of land cover. Therefore, the fuzziness of land cover types is due to the inherent continuity of nature, which leads to the imprecise definitions of land covers and sensor resolutions. It is more reasonable to describe the pixel in terms of membership value. A land cover object is actually a fuzzy spatial object.

Figure 1.2 Fuzziness of land covers

1.1.4 Importance of fuzzy spatial objects

Fuzzy spatial objects have become more and more important in GIS applications. When spatial phenomena are generalized by the crisp form, a lot of quantitative information is neglected. As Burrough (1996) pointed out, if soil types are represented as crisp objects with crisp boundaries, the transition from one type of soil to another is totally lost, which cannot reflect the reality. In reality, some concepts should not be considered as crisp objects at all (for example, mountains, oceans and the Yangtze River delta) since their boundaries are totally indeterminate. As we mentioned before, land cover is also a fuzzy spatial object. And land cover always changes gradually unless there is an abrupt change in nature. For example, forest will slowly degrade into bush when the natural environment degenerates. During the process of change, the land cover normally changes from forest, to mixed forest and bush, and finally to bush. If we crisply classify TM images into crisp land cover types, then the change from forest to bush appears to have happened suddenly. Mixed information cannot be reflected by crisp land cover objects.

1.1.5 Monitoring land cover changes in China

Land resource poses one of the biggest problems in China. Because of the growth in population and the economy, the contradiction between land resources and humans is becoming more and more severe. On the one hand, more arable land is necessary to feed more people. On the other hand, the growth of the economy accelerates urbanization, which always results in a decrease in cultivated land. According to the Chinese 21st Agenda, the cultivated land accounted for 13,248 million ha in 1985 but was down to 13,003 million ha in 1996, based on the results of a detailed land use investigation. During these 11 years, 244,000 ha of cultivated land were lost each year. Because of the severity of the problem, at the beginning of ‘80s, the Chinese government proposed a long-term strategy called “Treasuring and using reasonably every inch of land”. During these years, the problems such as the haphazard use of land and illegal land claims happened frequently. Now the Chinese government is practicing the strictest land management policy in the world. The core of the policy is the ruling that all proposed construction involving more than 0.2 ha must first be approved by the central government. Furthermore the Land Resource Ministry has proposed a practical policy to maintain cultivated land in dynamic balance. When cultivated land is used for other purposes, an equivalent amount of land for cultivation should be retrieved from other types of land.

To reach this goal, in 1999 China began a large-scale investigation project on land use and land cover change. The main purpose of the investigation is to monitor and understand LULC changes all over the country. In 2000, 87 cities were monitored using TM images and SPOT satellite images. The land cover is classified into eight categories, which can be subdivided into 55 sub-categories. The eight categories are cultivated land, water area, forest, orchard, industrial land, transportation land, pasture and unused land. In the project, land cover change is identified by using bi-temporal images and then verified by air-photos or by fieldwork. The identification of changes from satellite images was done manually. As a consequence many skilled people were involved in identifying the changes. The land use change analysis focuses on changes in the eight categories. GIS is utilized as a conventional tool to summarize the changes in land use types and sizes. It has been estimated that the project will last around 10 years.

1.2 Problem statement

1.2.1 Research motivation

The research motivation has been generalized from the practical problems. The goal of the above project is to investigate land use and land cover changes in China so that the government can introduce the correct land policies to achieve the dynamic balance of cultivated land. The common procedures for detecting change are as follows: obtain up-to-date TM or SPOT images and collect old TM or SPOT images; classify them crisply into several land use and land cover classes; compare the differences between two classified results by subtraction; and then interpret the differences manually. The procedure is outlined in Figure 1.3. Because of classification errors, considerable manual interpretation and fieldwork are needed to decide whether there has been a real change or not.

However, this process produces some errors when the change is not obvious, for example, there is a change that bush slowly degenerates into grassland. This procedure will report the change when there are these two classes in some areas. Actually in many cases the reality is that the number of trees is decreasing and the amount of grass is increasing. The change from one class to another is not so abrupt. There is only a degree of change from bush to grass. Therefore it is better for land use and land cover objects to be modeled as fuzzy spatial objects.

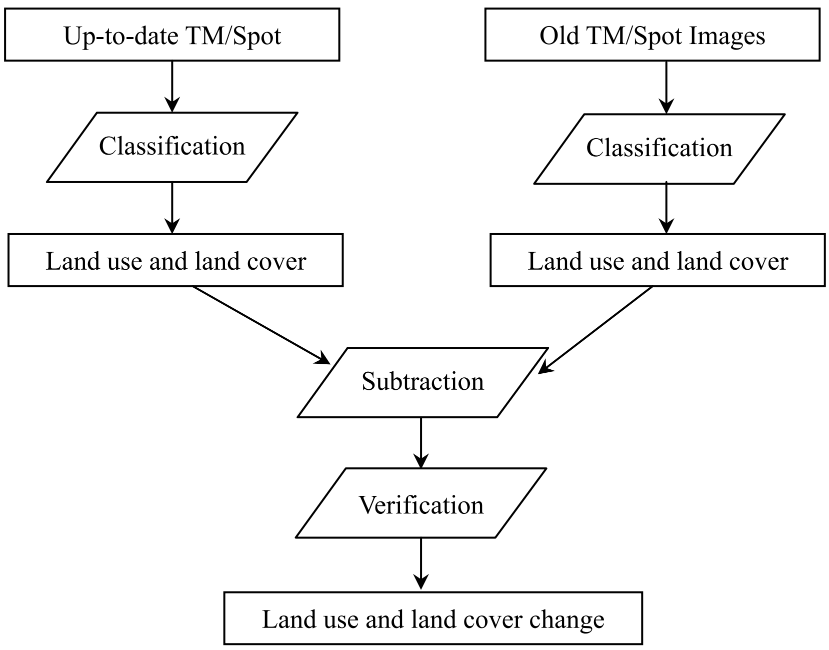


Figure 1.3 Procedures for detecting land cover changes in the project *"Monitoring land cover changes of China"*

1.2.2 Formal definition of spatial objects

The premise of data modeling is that the concepts of all objects should be clearly defined. When we start to model the objects with fuzzy characteristics, the first problem is how to define fuzzy spatial objects for GIS applications.

Crisp spatial objects have been well researched and formally defined in GIS. Point, line and polygon are three primitives in GIS. The method of defining a crisp point, line and polygon belongs to the field of geometry. The problems have been abstracted into topology theory. Intuitively speaking, topology deals with continuous deformations of objects. The most primitive concepts are open set and closed set, neighborhood, connectedness and so on (these will be introduced in Chapter 2). Crisp point, line and polygon have been formally defined in the crisp topological space. For example, a simple crisp region has been defined as a regular closed set whose interior is connected in the connected (crisp) topological space (Egenhofer and Franzosa 1991). Intuitively, the closure can be regarded as a closed disk; the interior is a disk without the boundary; and the boundary is the difference between the closure and its interior (Figure 1.4).

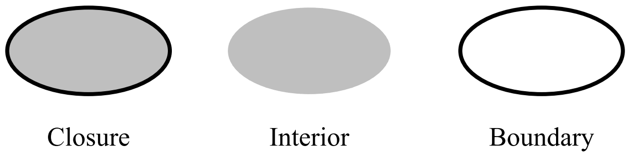


Figure 1.4 Closure, interior and boundary of a closed disk

A crisp spatial object is of course a crisp set, which has a characteristic function for its elements. The value of a characteristic function is either 1 or 0. The boundary of a crisp spatial object is also crisp. A piece of grassland can be represented by a regular closed set whose characteristic value is equal to 1 for its elements.

In order to handle fuzzy spatial objects, a formal definition of fuzzy point, fuzzy line and fuzzy region seems necessary in GIS. The problem is what a fuzzy point should look like. Assume a fuzzy object *A* whose membership function is as follows (Figure 1.5):

$$MV = \begin{cases} 1 & \text{where} \quad x = 0 \; y = 0 \\ (0,1) & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 = 1 \end{cases}$$

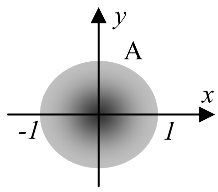


Figure 1.5 A fuzzy spatial object

The core of the object is a point, and the outer boundary is a line where all membership values of the line are zero. Is it a fuzzy point, a fuzzy region, or something else?

1.2.3 Topological relations

Another problem in modeling fuzzy spatial objects is the topological relations between fuzzy spatial objects. It is one of the most fundamental properties between spatial objects, since it can answer questions such as “who is my neighbor?”

The topological relations have been formally identified in the crisp topological space between crisp spatial objects. For example, Egenhofer and Franzosa (1991) and Egenhofer and Herring (1990a, 1990b) have introduced the 4-intersection and 9-intersection approaches in connected topological space by using the interiors, boundaries and exteriors between two crisp subsets. Eight relations have been identified between two simple regions in the two-dimensional Euclidean space R^2 (Figure 1.6).

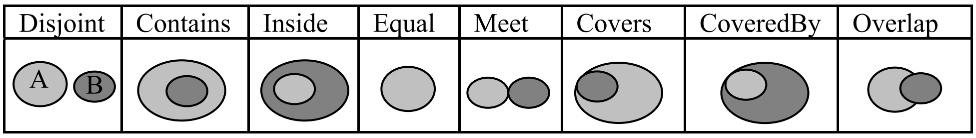


Figure 1.6 Eight topological relations between two simple regions in R^2

It is not clear what the topological relation is between two fuzzy spatial objects? For example, there are two fuzzy spatial objects. What is the topological relation between them? And how can it be identified (Figure 1.7)?

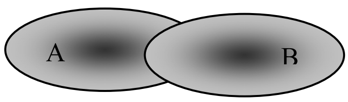


Figure 1.7 Two fuzzy spatial objects

1.2.4 Modeling spatial objects

It is well known now that GIS is a tool that can store, retrieve, analyze and display the information related to spatial data. The advantage of GIS lies in its powerful ability to analyze spatial objects. Spatial overlay and buffering, as well as conventional statistical analysis, can be easily done in almost any kind of GIS software package. GIS can be used not only in the traditional realms such as geography, geology, environment and natural resource management, but also in other fields where spatial data exist, such as medical science and social economics. All applications with spatial distribution can adopt GIS as a tool for analyzing their data. To date, GIS has been developed to such an extent that it reaches almost all corners of our society.

1.2.4.1 Conventional data models

The core of GIS is its data model. Two diametrically opposed geographic data models are nowadays available for encapsulating the aspects of interest of spatial phenomena.

These are the object-based model and the field-based model, according to Worboys (1995). In the object-based model (also called the feature-based model or vector-based model), natural entities or objects are represented by crisply defined primitives: points, lines and polygons.

Figure 1.8(II) depicts a simple vector data model for representing the geometric parts of points, lines and polygons, which are primitives of the model (Figure 1.8(I)). In the model, lines link a series of exactly known points, and polygons are bounded by exactly defined lines. Topological relations between these objects can be formalized in a cell complex structure based on algebraic topology.

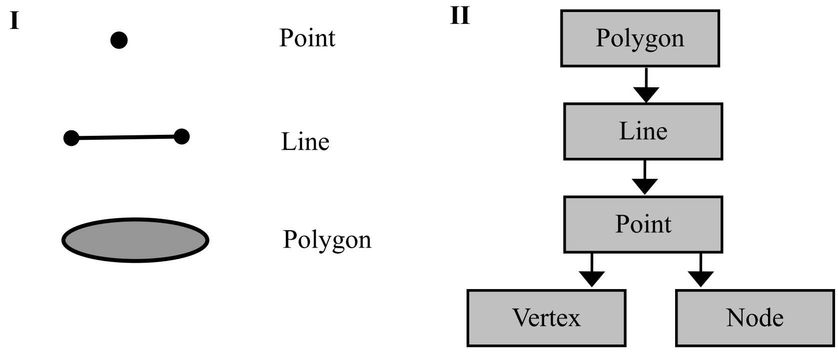


Figure 1.8 Vector data models for crisp spatial objects
I: Primitives, II: Vector data model

While the object-based model regards spatial data as point, line and polygon objects, the field-based model treats spatial data as a collection of spatial distributions, where each distribution may be formulated as a mathematical function from a spatial framework to an attribute domain. In practice, these attributes are often discretized to a grid at a given level of resolution. In this model the boundary of different objects is not formed. Figure 1.9(II) shows a common raster data model where the grid is represented by fixed-sized pixels. The topological relations are imbedded in the attribute representation and there is no explicit expression for the topological relations between each other.

These two models are the extreme abstractions of reality that are attractive for their logical consistency and their ease of handling using conventional reasoning and mathematics (Worboys 1995). The object model in its simplest form has been implemented using a relational database structure for the attributes. For the geometry and topological relations, there are well-designed data structures in GIS software. The continuous data in the field-based model can be handled in a purely relational database structure once it has been discretized to a regular grid. These models are nowadays extensively used in GIS. Figure 1.9 shows how to represent land cover objects by these two models.

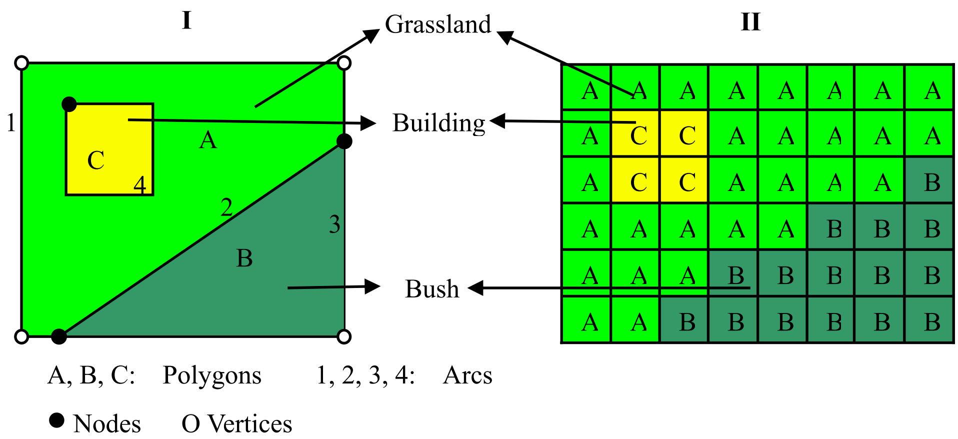


Figure 1.9 Two models for representing land cover objects
I: Vector data model, II: Raster data model

Figure 1.8 shows both models can represent crisp spatial objects. The advantage of the object data model is that the topological relations can be expressed explicitly; however, it lacks the ability to handle the continuity of objects. In Figure 1.9(I), the topological relations between polygons and lines can be expressed by A: 1, 2, 0, 4 (0 means that 4 is a hole of A); B: 2, 3; C: 4. The advantage of the raster data model lies in its capability to model the continuity of spatial objects; however, it lacks a definite form for expressing boundaries, since it implies planar coordinates and adopts only the attribute domain in the representation.

1.2.4.2 Modeling fuzzy spatial objects

With the extensive application and increased requirements of GIS, it is becoming more and more important to model fuzzy spatial objects. It is necessary to analyze whether the two conventional models are sufficient to represent fuzzy spatial objects and relationships, especially the topological relations between them. In the object data model, the boundary of a crisp spatial object is explicitly represented, and the topological relations can be generated directly, as mentioned before. The field data model can express some continuity of spatial objects; however, the topological relations are implicit. It can be perceived that both models have some difficulties in conveying fuzzy spatial objects. The object data model represents a crisp object by the boundary (and its label point). It implies that the attribute of this object should be identical within its boundary. An obvious characteristic of a fuzzy spatial object is that its membership values vary along with its location. Therefore, it is almost impossible to represent fuzzy spatial objects in the object data model. The field data model has the capability to depict the continuity of a spatial object. It usually represents one attribute within one pixel. And since there is no boundary, objects are not formed in this data model. However, in nature, it is very usual for one location to have several membership values that belong to different classes. For example, one pixel may have membership value 0.3 for grassland,

and 0.7 for bush. Therefore, several attribute domains should be designed to denote them. The topological relations can be generated only after several steps, including the derivation of fuzzy objects, the generation of boundaries, and then the identification of the topological relations. It is also inconvenient to depict fuzzy spatial objects, especially for topological relations between fuzzy spatial objects.

According to the above analysis, it can be perceived that both models have some difficulties in representing fuzzy spatial objects and their topological relations. It is necessary to create a model that is capable of representing fuzzy spatial objects efficiently, with a sound topological structure, as depicted in the vector data model. Ideally, the fuzzy polygon, fuzzy line and fuzzy point should be able to be modeled as shown in Figure 1.10.

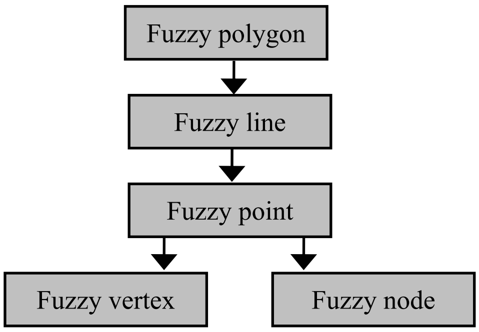


Figure 1.10 A model for fuzzy spatial objects

1.2.5 Modeling land cover changes

A model should be able to tackle some problems for GIS applications. The correctness of a theoretic fuzzy spatial model has to be verified in practice.

1.2.5.1 Generation of fuzzy land cover objects

The importance of land in China cannot be overemphasized. Concerning transitional land cover changes, it is better to model land covers as fuzzy objects as we suggested in Section 1.1. The key problem then is how to derive the membership values for land cover objects.

In general, membership values can be calculated by two kinds of methods: active and passive (Cheng *et al.* 1997). The active method derives the membership function and values by experts or based on some knowledge. For example, if we divide people into three fuzzy classes: “young”, “middle” and “old”, we can subjectively design membership values for these classes. The passive method calculates the fuzzy membership values according to the data itself. For example, we can do a survey for “young”, “middle” and “old”, and calculate the membership values based on the census.

Our question is how to generate membership values for fuzzy land cover objects. What is the procedure then? For example, there is a TM image (Figure 1.11) of an area. How can a fuzzy land cover object be generated from this image?

Figure 1.11 A TM image

1.2.5.2 Querying fuzzy land cover objects

One of the fundamental capabilities of GIS is to query spatial objects in different ways, especially its power to find spatial objects according to size, direction and topological relations. This thesis is particularly concerned with queries based on topological relations. The question then is how to query fuzzy objects based on topological relations to meet different kinds of requirements.

1.2.5.3 Reasoning about land cover changes

Reasoning is the process of combining facts and rules to deduce new facts (Sharma 1996). A reasoning system usually includes the following parts: database, knowledge acquisition and knowledge base, inference rules and inference machine (Figure 1.12).

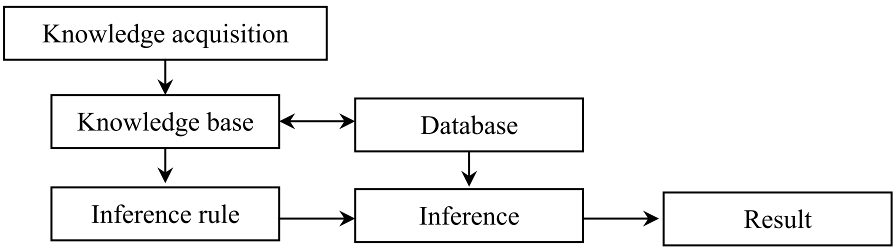


Figure 1.12 General reasoning procedure

For land cover objects, it is important to investigate the size of changes. Where are these changes? And is the change large, average or small?

1.3 Objectives

The objective of this research is to build a formal framework for modeling fuzzy spatial objects in GIS. It involves the theoretic modeling of fuzzy spatial objects and building a practical data model to solve some practical problems by using fuzzy spatial objects. The objective can be broken down into a theoretic part and a practical part. The theoretic objectives are:

(1) To derive a formal method for defining fuzzy spatial objects;
(2) To identify topological relations between fuzzy spatial objects; and
(3) To propose a formal framework for modeling fuzzy spatial objects.

Practically, it is necessary to construct a concrete fuzzy spatial object model with the formal framework and solve some problems for certain applications. Because of the importance of land cover and its change in China, modeling land cover objects is selected as a practical application. The practical objectives are:

(1) To generate fuzzy land cover objects;
(2) To query fuzzy land cover objects according to topological relations; and
(3) To apply fuzzy spatial objects for reasoning about land cover changes.

1.4 Research questions

In order to achieve the above objectives, the problems stated in Section 1.2 can be further broken down into the research questions below:

(1) What are fuzzy points, fuzzy lines and fuzzy regions? How can we derive a method to define them mathematically? What is the difference between fuzzy spatial objects and crisp spatial objects?
(2) What are the topological relations between fuzzy spatial objects? How can we propose a formal approach to formalizing topological relations between fuzzy spatial objects? How can we identify the topological relations between fuzzy spatial objects? For example, how can we identify topological relations between two fuzzy regions? What are they? If we compare approaches to identifying topological relations between crisp spatial objects, what are the differences?
(3) How can we derive a structure for modeling fuzzy points, fuzzy lines and fuzzy regions?
(4) How can we derive a method for calculating the membership values for fuzzy land cover objects?
(5) How can we query fuzzy spatial objects based on various topological relations?
(6) How can we infer the change size based on fuzzy land cover objects?

1.5 Methodology

In order to define the concepts of fuzzy points, fuzzy lines and fuzzy regions formally, an appropriate theory is the premise of the research. It is noticed that topology theory is adopted for defining crisp points, crisp lines and crisp regions. The newly proposed fuzzy topology theory should be sound for defining fuzzy points, fuzzy lines and fuzzy regions since it is constructed based on fuzzy sets.

The topological relations between crisp spatial objects are formalized and identified based on the crisp topological space. The cell complex structure is widely applied to represent crisp points, lines and polygons, and to express the topological relations between them. Therefore, it is also appropriate to turn to a fuzzy topological space to formalize topological relations between fuzzy spatial objects, and for a structure to model fuzzy points, lines and regions.

In order to model fuzzy land cover objects, a method for generating membership values should be proposed. Fuzzy set theory is the ideal tool for deriving membership values of fuzzy land cover objects. For the same reason, the fuzzy reasoning method is adopted to infer the land cover changes based on the fuzzy spatial objects.

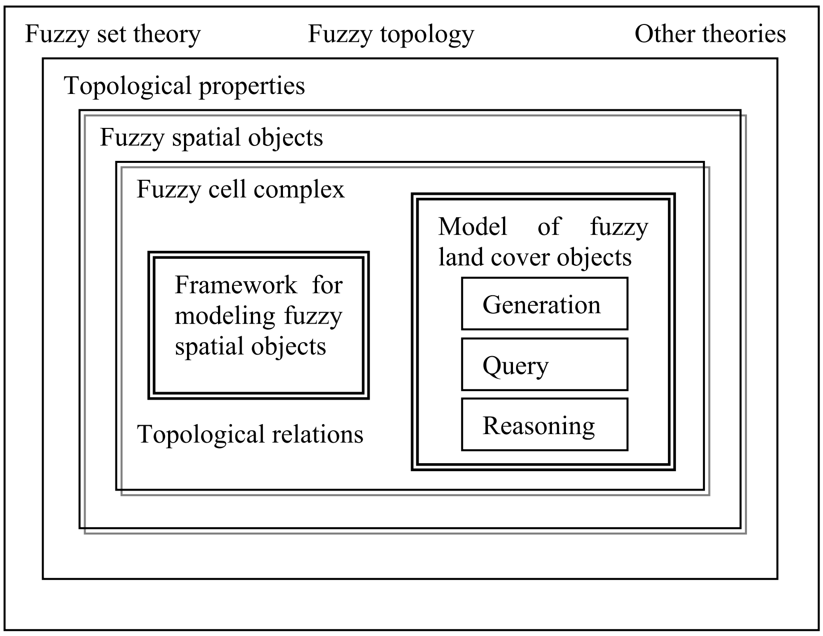


Figure 1.13 Research methodology

The research starts with the basic concepts of fuzzy set theory, topology, fuzzy topology and fuzzy reasoning. Based on the analysis of topological properties of fuzzy set in fuzzy topological space, a formal method is proposed for formalizing topological relations between fuzzy regions in a special fuzzy topological space. The topological properties are then further extended into some novel concepts. These concepts are then adopted to define a fuzzy region and to formalize the topological relations between fuzzy regions. In order to define fuzzy points and fuzzy lines and to identify topological relations between them, a fuzzy cell complex structure is proposed, like a crisp cell structure for crisp spatial objects. This fuzzy structure constitutes the formal framework for modeling fuzzy spatial objects. And it is then applied in modeling fuzzy land cover objects, which includes generating fuzzy land cover objects, querying these objects based on topological relations, and reasoning about changes of land covers. Sanya city, which is located on Hainan Island in China, is selected for the practical application since many changes have happened since 1992. The detailed situation in Sanya city is introduced in Chapter 6. The methodology is structured in Figure 1.13.

The research procedure is sketched in Figure 1.14.

1.6 Structure of the thesis

This research handles two aspects: a formal framework for modeling fuzzy spatial objects and the practical modeling fuzzy land cover objects. Chapter 2 introduces the basic theory that will be adopted in the following chapters. Chapters 3, 4 and 5 discuss the framework for modeling fuzzy spatial objects and topological relations theoretically. Chapters 6, 7 and 8 cover applications of modeling fuzzy land cover objects: from generating these objects, to querying them based on various topological relations, to using a fuzzy reasoning approach for reasoning about the changes. Conclusions and discussions are summarized in Chapter 9. The main contents of the thesis can be described as follows:

Chapter 1 gives an overview of the thesis, including the background of the research, research problems, objectives, research questions and methodology. The background introduces the following aspects: what fuzzy spatial objects intuitively are and why we need them, what land cover objects are and why they are fuzzy, and the importance of adopting fuzzy spatial objects to analyze land cover changes in China. The problem statements expound the main research interests, such as fuzzy spatial objects, topological relations, formal fuzzy spatial data model, and modeling fuzzy land cover objects. The objectives and key research questions are put forward based on the problem statements. The methodology describes how to achieve the objectives and solve the research questions.

Chapter 2 explains some fundamental concepts of fuzzy set theory, topology and fuzzy topology, and fuzzy reasoning, which are the main tools of the research.

Chapter 3 discusses the topological relations between fuzzy spatial objects. It starts with the definitions of fuzzy boundary defined in fuzzy topological space. Then the limitations of the 9-intersection matrix in a general fuzzy topological space are explained. In order to formalize the topological relations between fuzzy spatial objects, a special fuzzy topological space is defined, in which a 3*3-intersection matrix can be defined, which is the same form as the 9-intersection. Furthermore, a 4*4-intersection matrix is formalized in this fuzzy topological space, and more topological relations are identified between two simple fuzzy regions defined in this special fuzzy topological space.

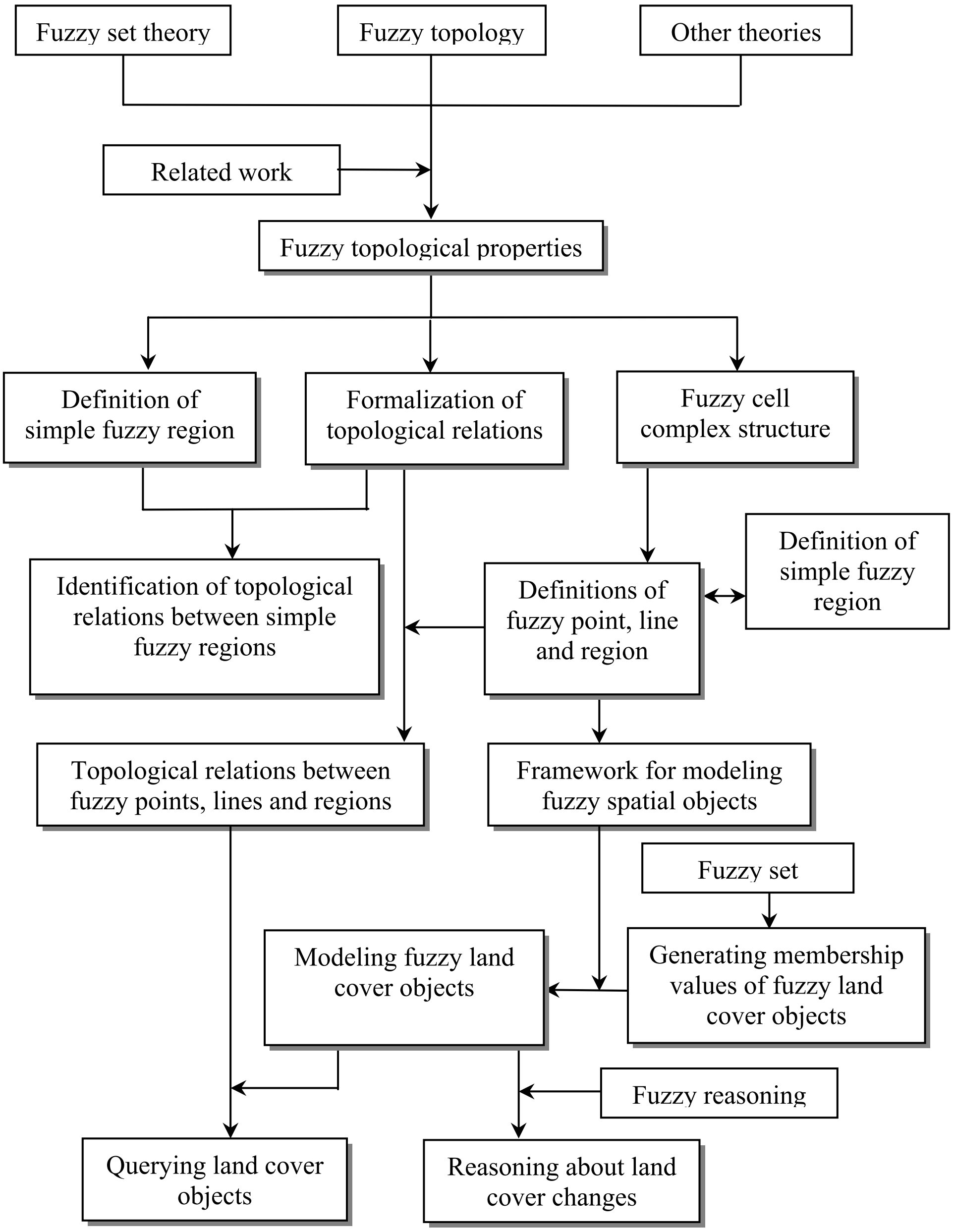


Figure 1.14 Research procedures

Chapter 4 discusses another method of identifying the topological relations between simple fuzzy regions. Several novel notions in general fuzzy topological space are proposed. Based on these concepts, two novel forms of a 3*3-intersection matrix and one 4*4-intersection matrix are formalized. A formal definition of a simple fuzzy region in a general fuzzy topological space is then given. The topological relations are also identified between simple fuzzy regions. A comparison is then given between this method and the method adopted in Chapter 3.

Chapter 5 proposes a fuzzy cell complex structure, which constitutes a theoretic framework for modeling different kinds of fuzzy spatial objects, including fuzzy regions, fuzzy lines and fuzzy points. The fuzzy regions, lines and points are defined and the topological relations between the fuzzy regions, lines and points are identified by using the 3*3-intersection matrix on this structure.

Chapter 6 discusses the general procedure for generating fuzzy spatial objects. After that, a composite method is proposed for generating fuzzy land cover objects from TM images.

Chapter 7 proposes four methods for querying fuzzy spatial objects based on different topological relations and different requirements. The differences between these methods are compared.

Chapter 8 discusses a method of applying fuzzy spatial objects for analysis. The land cover objects are compared and their changes are inferred using a fuzzy reasoning method. The result shows that adopting fuzzy land cover objects rather than crisp land covers produces more accurate results.

Chapter 9 presents conclusions and discussions, which point out the main contribution of this thesis and further research questions.

Chapter Two

Fuzzy Set Theory and Topology

Spatial objects can be regarded as sets from a set theoretic point of view. Crisp (non-fuzzy) spatial objects have been well defined based on classical (crisp) set theory and general (crisp) topology. In order to define and model fuzzy spatial objects in GIS, it is necessary to investigate their essence using fuzzy set theory and fuzzy topology. This chapter will review some basic concepts in classical set theory, fuzzy set theory, general topology and fuzzy topology, which will be adopted for fuzzy spatial object definition and modeling in the following chapters.

2.1 Classical set theory

2.1.1 Set

A *set* is a collection of well-distinguishable objects (Kainz 2004). Any object in the collection is an *element* or a *member* of the set. An element x of a set X is written as $x \in X$. If x is not a member of X we write $x \notin X$. If a set has a finite number of elements, we call it a *finite set*, otherwise we call it *infinite*. A set with no elements is called the *empty set* and is denoted as $\varnothing$.

A finite set can be specified explicitly by listing all its elements. For example, a set A consisting of the natural numbers smaller than 4 can be listed as $A = \{1,2,3\}$. A set can also be described implicitly by means of a free variable x in a predicate $P(x)$. The set A can be described by the properties $A = \{x \in N : x < 4\}$; N is the set of all natural numbers. The *cardinality* of a set is the number of its elements.

If each element of a set A is an element of a set B, then A is a *subset* of B, written as $A \subseteq B$. The *universe of discourse* (or *universe*) is the set consisting of all elements in a certain study. If U is the universe, then every subset $A \subseteq U$. A is called a *proper subset* of B when $A \subseteq B$ and $A \neq B$. $\varnothing \subseteq A$. The set of all subsets of a set A is called the *power set* of A, denoted as $\wp(A)$, for example, the power set of the set $A = \{1,2,3\}$ is

$\wp(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$. We can also draw a set with a VENN diagram (Figure 2.1)

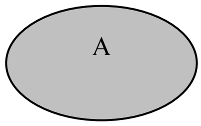


Figure 2.1 VENN diagram

2.1.2 Set operations

Four operations are often used between sets in set theory: union, intersection, difference and complement.

Union: the union of two sets *A* and *B*, written as $A \cup B$, is the set $A \cup B = \{x : x \in A \; or \; x \in B\}$. It is the set containing all the elements that belongs either to *A* or to *B*, or to both.

Intersection: the intersection of two sets *A* and *B*, written as $A \cap B$, is the set $A \cap B = \{x : x \in A \; and \; x \in B\}$. *A* and *B* are *disjoint* if $A \cap B = \varnothing$.

Union and intersection can be defined for more than two sets. Let *I* be an arbitrary finite or infinite set. Every element $i \in I$ is assigned a set A_i, then the union of A_i is defined as $\bigcup A_i = \{x : \exists i \in I[x \in A_i]\}$, where $\exists i$ stands for *there exists i*. The intersection of A_i is $\bigcap A_i = \{x : \forall i \in I[x \in A_i]\}$, where $\forall i$ means *for all i*.

Difference: the difference between two sets *A* and *B*, written as $A - B$, is the set $A - B = \{x : x \in A \; and \; x \notin B\}$.

Complement: the complement of a set *A* in the universe *U*, written as A^c, is the set $A^c = U - A = \{x : x \notin A\}$.

2.1.3 Cartesian product and relation

The *Cartesian product* of two sets *A* and *B*, written as $A \times B$, is the set of all pairs $A \times B = \{< a,b >: a \in A \; and \; b \in B\}$. A *binary relation R* over $A \times B$ is a subset of $A \times B$. The set *A* is called the *domain* of *R* and *B* is called the *codomain*. We can write $< a,b > \in R$ also as aRb, or $R(a,b)$. If the relation is defined over $A \times A$, we call it a *relation on A*.

Let *R* be a relation over $A \times B$, the *inverse relation* (or *reverse*) R^{-1} is defined as the relation over $B \times A$ such that $R^{-1} = \{< b,a >:< a,b > \in R\}$. Let *R* be a relation on *A*. It may have some of the following properties:

(1) R is *reflexive* if xRx for every x in A;
(2) R is *irreflexive* if xRx for no x in A;
(3) R is *symmetric* if xRy implies yRx for every x, y in A;
(4) R is *antisymmetric* if xRy and yRx together imply $x = y$ for every x, y in A;
(5) R is *transitive* if xRy and yRz together imply xRz for every x, y, z in A.

A reflexive, symmetric and transitive relation is called an *equivalence relation*. A reflexive, antisymmetric and transitive relation is called an *order relation*. A set equipped with an order relation is called a *partially ordered set*, or *poset* for short.

Let P be a poset, and $A \subseteq P$. An element $b \in P$ is called an *upper bound* of A if $a \leq b$ for every $a \in A$. $b \in P$ is called a *lower bound* of A if $a \geq b$ for every $a \in A$. An element $b \in P$ is called the *join* (or *the least upper bound*, or *supremum*) of A, denoted by $\sup A$, or $\bigvee A$ if b is an upper bound of A and if a is an upper bound of A, then $b \leq a$. $b \in P$ is called the *meet* (or *the greatest lower bound* or *infimum*) of A, denoted by $\inf A$, or $\bigwedge A$, if b is a lower bound of A and if a is a lower bound of A, then $b \geq a$.

A new relation can be generated by composing a sequence of relations. Let R_1 be a relation from A to B, and R_2 be a relation from B to C. The *composite relation* from A to C, written as R_1R_2, is defined as:

$$R_1R_2 = \{<a,c>: a \in A \text{ and } c \in C \text{ and } \exists b[b \in B \text{ and } <a,b> \in R_1 \text{ and } <b,c> \in R_2]\}$$

2.1.4 Mappings

Mappings (functions) are special kinds of binary relations. A *mapping f* from A to B, written as $f : A \rightarrow B$, is a binary relation from A to B such that for every $a \in A$, there exists a unique $b \in B$ such that $<a,b> \in f$, and is written as $f(a) = b$. A is called the *domain*, and B is called the *codomain* (or *image*) of *f*. A mapping from A to B is called *surjective* (*onto* or *surjection*) if $f(A) = B$. A mapping from A to B is called *injective* (*one-to-one,* or *injection*) if $a \neq a'$, then $f(a) \neq f(a')$. A mapping from A to B is called *bijective* (*one-to-one and onto,*or *bijection*) if it is both surjective and injective.

2.2 General topology

Topology is a central concept in every GIS. Intuitively speaking, it deals with the structural representation of spatial features and their properties that remain invariant under certain transformations (Kainz 2004).

2.2.1 Topological space

Mathematically, a topology is a collection of subsets on a set that follows certain rules. Let *X* be a set, and τ be a collection of subsets of *X*, *i.e.*, $\tau \subseteq \wp(X)$ (Bredon 1991). If

$$\varnothing \in \tau, X \in \tau$$
$$\forall A, B \in \tau, A \cap B \in \tau$$
$$\forall A_i \in \tau, \bigcup_{i \in I} A_i \in \tau$$

Then τ is called a *topology* on *X*. (X, τ) is called a *topological space* on *X*. In order to differentiate with fuzzy topological space (which will be introduced later), we call it *crisp* (or *ordinary*) *topological space,* or *cts* for short. Every element $x \in X$ is called a *point* of the topological space. Every element of τ is called an *open set* of the topological space (X, τ). A set is *closed* if its complement is open.

2.2.2 Interior, closure, boundary and exterior

For GIS data modeling, the most important concepts are the interior and the closure of a subset in the topological space. Let *X* be a topological space. The union of all open sets contained in subset *A* is called the *interior* of *A*, denoted by A^o. A^o is the largest open set contained in *A*. The intersection of all the closed sets containing *A* is called the *closure* of *A*, denoted by A^-. A^- is the smallest closed set containing *A*. A subset *A* in *X* is called a *neighborhood* of point *x* if there exists a $x \in B$ such that $x \in B \subseteq A$. The union of all neighborhoods of a point is called the *neighborhood system* of that point. A subset is open iff (if and only if) it is a neighborhood of each of its points. The topology on a set can also be defined by using the neighborhood system.

We are also concerned about the boundary and the exterior of a subset in the topological space. In a topological space *X*, the *boundary* of a subset *A* is defined as the difference between the closure and the interior of the subset *A*, *i.e.*, $\partial A = A^- - A^o$. The *exterior* of *A* is the complement of A^- and denoted by A^e. Obviously A^e is an open set.

The following properties hold for the interior, the boundary, the closure and the exterior of the subset(s) in the crisp topological space.

Proposition 2.1 *Let A, B be two subsets of cts X.*

(1) $A^o \subseteq A$, $(A^o)^o = A^o$;

(2) $A \subseteq B \Rightarrow A^o \subseteq B^o$;

(3) $(A \cap B)^o = A^o \cap B^o$, $(A \cup B)^o \supseteq A^o \cup B^o$.

Proposition 2.2 *Let A, B be two subsets of cts X.*

(1) $A \subseteq A^-, (A^-)^- = A^-$;

(2) $A \subseteq B \Rightarrow A^- \subseteq B^-$;

(3) $(A \cap B)^- \subseteq A^- \cap B^-$, $(A \cup B)^- = A^- \cup B^-$.

Proposition 2.3 *Let A be a subset of cts X.* $A^{oc} = A^{c-}$, $A^{-c} = A^{co}$.

Proposition 2.4 *Let A be a subset of cts X.* $X = A^{o} \cup A^{oc} = A^{-} \cup A^{-c}$.

Proposition 2.5 *Let A be a subset of cts X.*

(1) $\partial A = A^{-} - A^{o} = A^{-} \cap A^{c-} = \partial(A^{c})$;

(2) $A^{-} = A^{o} \cup \partial A$;

(3) $X = A^{o} \cup \partial A \cup A^{e}$.

Proposition 2.6 *Let A be a subset of cts X.* A^{o}, ∂A, A^{e} *are mutually disjoint in X.*

The concept of the interior, the boundary and the closure can be illustrated by an example. Let $X = \{a,b,c,d\}$, $\tau = \{X, \varnothing, \{a\}, \{a,b\}\}$. τ is a topology on *X*. Suppose $A = \{a,b,c\}$, then $A^{o} = \{a,b\}$, $A^{-} = \{a,b,c,d\}$, $\partial A = \{c,d\} = A^{-} - A^{o}$, $A^{e} = \varnothing$, $A^{c} = \{d\}$, $A^{co} = \varnothing = A^{-c}$, $A^{oc} = \{c,d\} = A^{c-}$, and $\partial(A^{c}) = \{c,d\}$. It can be checked that the above propositions hold in this topological space.

2.2.3 T_0, T_1, Hausdorff, regular and normal space

A topological space is called a T_0 *space* if at least one of two distinct points has a neighborhood that does not contain the other point. A topological space is called a T_1 *space* if two distinct points have neighborhoods that do not contain the other point. A topological space *X* is called a *Hausdorff space* or T_2 if two distinct points $a, b \in X$ possess disjoint open neighborhoods, *i.e.*, there exist two open sets *A* and *B* with $a \in A$, $b \in B$ and $A \cap B = \varnothing$. A topological space is called a *regular space* or T_3 if it is T_1 and for every closed set *C* and every point *x* outside *C* there exists an open set *A* that contains *C* and a disjoint neighborhood *N* of *x*. A topological space is called a *normal space* or T_4 space if it is T_1 and for any two disjoint closed sets C_1 and C_2 there exist disjoint neighborhoods that contain the closed sets.

Every normal space is regular. Every regular space is a Hausdorff space, every Hausdorff space is a T_1 space, and every T_1 space is T_0.

2.2.4 Separation and connectedness

Two sets *A* and *B* in a topological space *X* are called *separated* if there exist two open (or closed) sets *H* and *K* such that $H \supseteq A$, $K \supseteq B$ and $H \cap B = \varnothing$, $A \cap K = \varnothing$. A topological space is *connected* if whenever it is represented as the union of two non-empty subsets $X = A \cup B$, then $A^{-} \cap B \neq \varnothing$, or $A \cap B^{-} \neq \varnothing$. Iff *X* is a connected space, then the only subsets of *X* that are both open and closed are the empty set and *X* itself; in other words, *X* cannot be represented as the union of two disjoint non-empty open (or closed) sets.

2.2.5 Homeomorphism and topological relation

We can define mappings between topological spaces. Let $f: X \rightarrow Y$ be a mapping from topological space *X* to topological space *Y*. The mapping *f* is called *continuous at point x* if, for every open set *B* containing $f(x)$, there is an open set *A* containing *x* such that the image of *A* is a subset of *B*, *i.e.*, $f(A) \subseteq B$. If *f* is continuous at every point of *X*, then *f* is called a *continuous* function.

If a mapping from topological space *X* to topological space *Y* is continuous, bijective, and its inverse is also continuous, then the mapping is called a *homeomorphism* (or *topological mapping*). A property of a topological space that is preserved by a homeomorphism is called a *topological property* or a *topological invariant*.

Let *R* be a binary relation from subset $A \subseteq X$ to subset $B \subseteq X$ on topological space *X*. *R* is called a *topological relation from A to B on X* if *R* is a topological invariant, *i.e.*, *R* is a topological property under a topological mapping from topological space *X* to topological space *Y*.

2.2.6 Metric and Euclidean space

Let *X* be a non-empty set and *d* a mapping $X \times X \rightarrow R_0^+$ such that for every $x, y \in X$:

(1) $d(x, y) = 0$ iff $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) + d(y, z) \geq d(x, z)$.

Then the pair (X, d) is called a *metric space* and *d* is called a *distance function (or a metric*) on *X*. Consider the real plane R^2, the metric d_E is called the *Euclidean distance* if $d_E(a,b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ between two points $a = (x_1, y_1), b = (x_2, y_2)$. The Euclidean distance is the shortest distance between two points. (R^2, d_E) is called the *two-dimensional Euclidean space*. This space is the usual space of plane geometry. We simplify it by R^2. (R^n, d_E) where $d_E(a,b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2}$ between two points $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n)$ is called an *n-dimensional Euclidean space,* and it is simplified by R^n.

In the metric space (X, d), define a set $N(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ $(\varepsilon \in R_0^+)$ as an open set. Then (X, d) is a topological space induced by the metric *d*. $N(x, \varepsilon)$ is a neighborhood of *x*. This topology is called a *metric topology* on *X*, and the space

(X, d) is called the *metric topological space*.

As a special case of metric space, the two-dimensional Euclidean space (R^2, d_E) is a topological space. The topology induced by the Euclidean distance is called the *Euclidean topology* or *usual topology*. In Euclidean space R^2, a neighborhood with radius $r < \varepsilon$ $(\varepsilon \in R_0^+)$ around a point is called an *open disk*, which is an open set. A neighborhood with radius $r \leq \varepsilon$ around a point is called a *closed disk*. A closed disk is a closed set. In R^2 the set of point *x* where radius $r = \varepsilon$ is the boundary of the open disk of point *x*. It is also called the *sphere.*

The Euclidean space is a special metric space. A metric space with metric topology is a normal space, and thus is a Hausdorff space. The Euclidean space is connected. In the Euclidean space, the interior of a closed disk $D(x, \varepsilon) = \{y \in R : d(x, y) \leq \varepsilon\}$ is $D^o(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, and its boundary is $\partial D(x, \varepsilon) = \{y \in X : d(x, y) = \varepsilon\}$. Figure 2.2 shows a closed disk, an open disk and the boundary of an open disk and a closed disk in R^2.

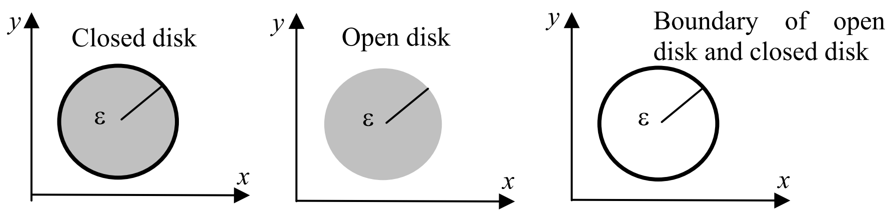


Figure 2.2 Closed disk, open disk and their boundary in R^2

2.2.7 Relationships between topological spaces

We have mentioned several topological spaces, such as T_0, T_1, T_2, T_3, T_4, connected space, metric space and Euclidean space. The relationships between these topological spaces are illustrated in Figure 2.3, *i.e.*, the Euclidean space is a metric space; a metric space is a normal space, etc. The Euclidean space is connected, while the others might not be connected.

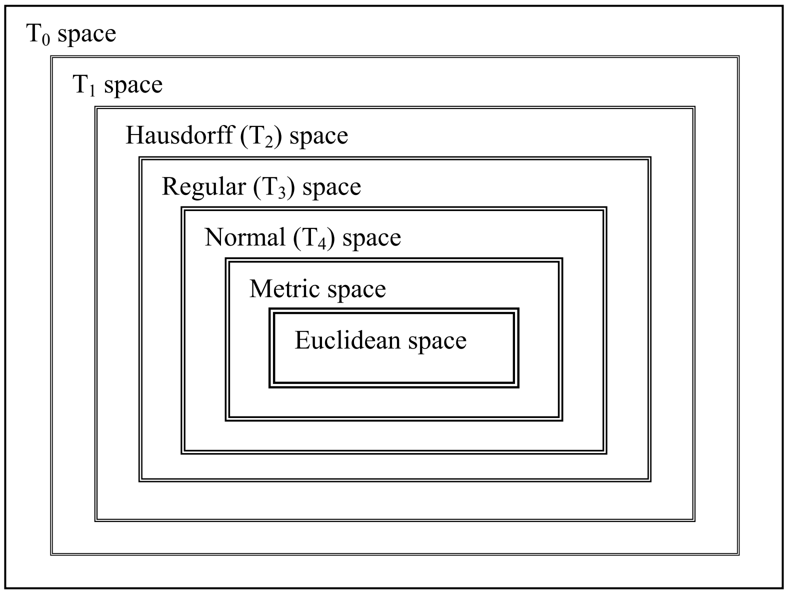


Figure 2.3 Relationships between topological spaces

2.3 Fuzzy set theory

2.3.1 Fuzzy set

Fuzzy set theory is the extension of classical set theory by allowing the membership of elements to range from 0 to 1. Let *X* be the universe of a classical set of objects. Membership in a classical subset *A* of *X* is often viewed as a characteristic function $\mu_A(x)$ (*x* is a generic element of *X*) from *X* to $\{0,1\}$ (Dubois and Prade 1980). $\{0,1\}$ is called a valuation set. If the valuation set is allowed to be the real interval $[0,1]$, *A* is called a *fuzzy set*. $\mu_A(x)$ is the *membership value* (or *degree of membership*) of *x* in *A*. Clearly, *A* is a subset of *X* that has no sharp boundary. A fuzzy set *A* can be represented by the set of pairs $A = \{(x, \mu_A(x)), x \in X\}$. When *X* is a finite set $X = \{x_1, x_2, ..., x_n\}$, then *A* can be expressed as $A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i$. When *X* is infinite, we write $A = \int_X \mu_A(x)/x$. We simplify the representation of a fuzzy set *A* by the supremum of the membership values $A(x)$ at every element *x* of *A* if there is no confusion.

According to the definition of a fuzzy set, the membership values with a range [0,1] have replaced the characteristic values of a crisp (or non-fuzzy) set in the universe. In this sense, the crisp set can be viewed as a special fuzzy set. Two fuzzy sets *A* and *B* are said to be *equal* if $\forall x \in X, A(x) = B(x)$, and denoted as $A = B$ (Figure 2.4). Fuzzy set A is said to be *contained* in B if $\forall x \in X, A(x) \le B(x)$. We write $A \subseteq B$ (Figure 2.4). When the inequality is strict, the inclusion is said to be *strict* and is denoted as $A \subset B$.

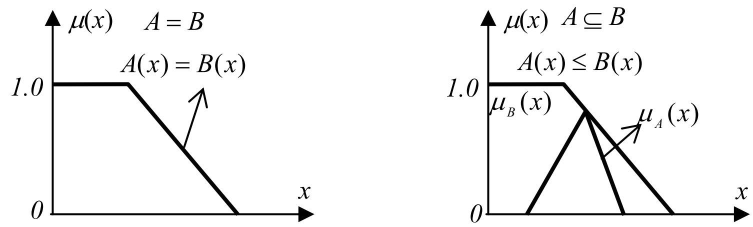


Figure 2.4 Fuzzy sets $A = B$ and $A \subseteq B$

The set of all fuzzy subsets of a set X is called the *fuzzy power set* of X, and we write it as $\tilde{\wp}(X)$. We also write it as $\wp(X)$ if there is no confusion with the crisp power set of X. The *support* of a fuzzy set A is the elements of x whose membership value is greater than zero, *i.e.*, $\text{supp}(A) = \{x \in X : \mu_A(x) > 0\}$. The elements of x such that $\mu_A(x) = 1/2$ are the *crossover points* of A. The *height* of a fuzzy set A is the largest membership value of A. A is said to be *normalized* if $\exists x \in X, \mu_A(x) = 1$.

As an example, let X be the set of natural numbers N; A be a fuzzy set of integers approximately equal to 10, defined as $A = 0.2/8 + 0.6/9 + 1.0/10 + 0.6/11 + 0.2/12$. The support of A is $\text{supp}(A) = \{8,9,10,11,12\}$. The height of A is $hgt(A) = 1$. It is a normalized fuzzy set.

2.3.2 Basic set-theoretic operations

The classical union, intersection and complement operations of two (crisp) subsets of X can be extended between fuzzy sets A and B of X as follows (Figure 2.5):

(1) *Union*:
$A \cup B = \{\max(A(x), B(x)) : \forall x \in X\}$, *i.e.*, $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$;

(2) *Intersection*:
$A \cap B = \{\min(A(x), B(x)) : \forall x \in X\}$, *i.e.*, $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$;

(3) *Complement:* the complement A^c of A is defined as:
$A^c = \{A^c(x) : A^c(x) = 1 - A(x), \forall x \in X\}$, *i.e.*, $\mu_{A^c}(x) = 1 - \mu_A(x)$.

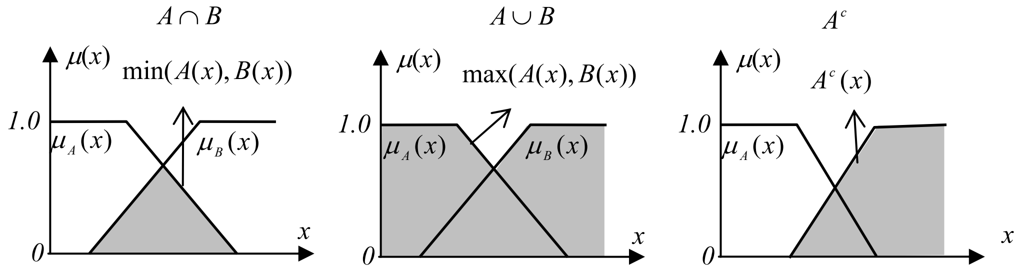


Figure 2.5 Intersection, union and complement of fuzzy sets

The following properties hold for fuzzy sets:

(1) *Commutativity:* $A \cup B = B \cup A$, $A \cap B = B \cap A$;
(2) *Associativity:* $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$;
(3) *Idempotency:* $A \cup A = A$, $A \cap A = A$;
(4) *Distributivity:*
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(5) *Absorption:* $A \cup \varnothing = A$, $A \cap X = A$;
(6) *De Morgan's Law:* $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$;
(7) *Involution:* $A^{cc} = A$;
(8) *Equivalence formula:* $(A^c \cup B) \cap (A \cup B^c) = (A^c \cap B^c) \cup (A \cap B)$;
(9) *Symmetrical difference formula:*
$(A^c \cap B) \cup (A \cap B^c) = (A^c \cup B^c) \cap (A \cup B)$.

However, the excluded-middle law is no longer true: $A \cup A^c \neq X$, $A \cap A^c \neq \varnothing$.

2.3.3 Extended operations

Because of the richness of the framework of fuzzy sets, more operations can be defined between fuzzy sets *A* and *B* on the fuzzy power set $\wp(X)$.

The differences between fuzzy sets *A* and *B* have several extensions. Three are often used in literatures.

(1) *Difference* $(A - B)$: $A - B = A \cap B^c$;
(2) *Bounded difference* $(A \nabla B)$: $A \nabla B = \max(0, A(x) - B(x))\ \forall x \in X$;
(3) *Absolute difference* $(A |-| B)$: $A |-| B = | A(x) - B(x) |\ \forall x \in X$.

Other useful operations are:

(4) *Product* $(A \cdot B)$: $A \cdot B = A(x) \cdot B(x)\ \ \forall x \in X$;
(5) *Bold intersection* $(A \underset{\bullet}{\cap} B)$: $(A \underset{\bullet}{\cap} B) = \max(0, A(x) + B(x) - 1))\ \ \forall x \in X$;
(6) *Probabilistic sum* $(A \hat{+} B)$: $A \hat{+} B = A(x) + B(x) - A(x) \cdot B(x)\ \ \forall x \in X$;

(7) *Bounded union* $(A \dot{\cup} B)$: $A \dot{\cup} B = \min(1, A(x) + B(x)) \quad \forall x \in X$.

These operations are illustrated in Figure 2.6.

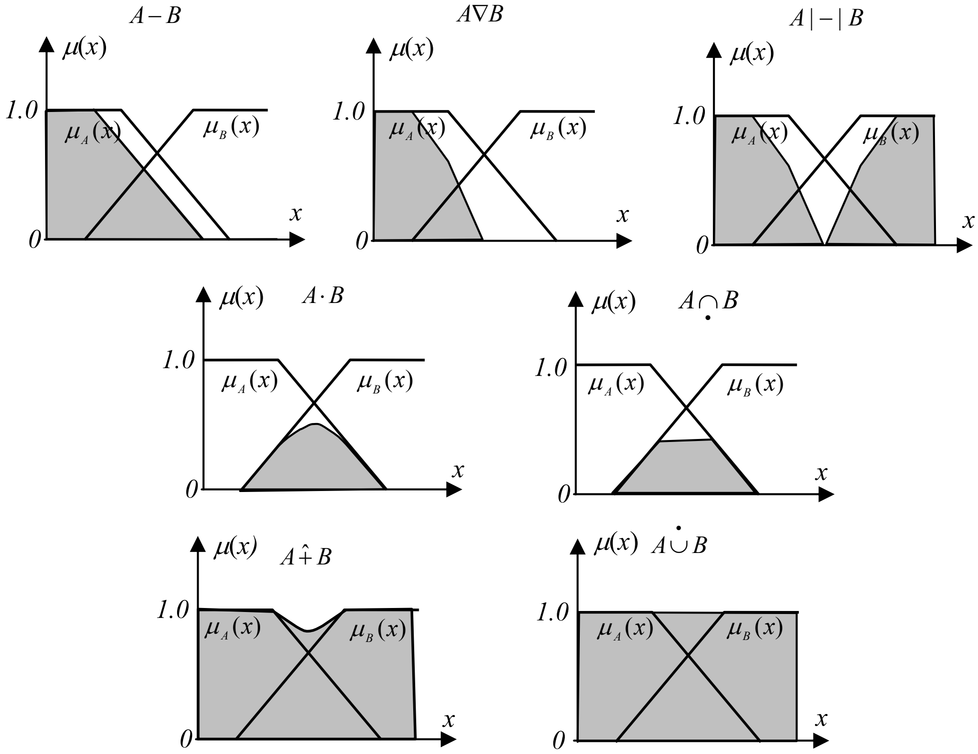


Figure 2.6 Difference, bounded difference, absolute difference, product, bold intersection, probabilistic sum and bounded union

More generally, most of these operators can be classified into two norms. A *T-norm* is a binary function from $[0,1] \times [0,1]$ to $[0,1]$ that satisfies the following conditions:

(1) $T(0,0) = 0$, $T(a,1) = T(1,a) = a$;

(2) $T(a,b) \le T(c,d)$ whenever $a \le c, b \le d$;

(3) $T(a,b) = T(b,a)$;

(4) $T(T(a,b),c) = T(a,T(b,c))$.

An *S-norm* is a binary function from $[0,1] \times [0,1]$ to $[0,1]$ that satisfies the following conditions:

(1) $S(1,1) = 1$ $S(a,0) = S(0,a) = a$;

(2) $S(a,b) \le S(c,d)$ whenever $a \le c, b \le d$;

(3) $S(a,b) = S(b,a)$;

(4) $S(S(a,b),c) = S(a,S(b,c))$.

In the above operators, $A \cap B$, $A \cdot B$, and $A \underset{\bullet}{\cap} B$ are *T*-norms.

2.3.4 α-cuts and fuzzy α-cuts

A (crisp) set of elements $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\} = \{x \in X : A(x) \geq \alpha\}$ of a fuzzy set *A* is called the *α-cut* (or *α-level*) A_α of *A*. The (crisp) set $A_{\bar{\alpha}} = \{x \in X : \mu_A(x) > \alpha\}$ is called the *strong* (or *strict*) *α-cut* of *A*. A fuzzy set *A* can be expressed in terms of the characteristic functions of its *α*-cuts according to the formula:

$A_\alpha = \sup_{\alpha \in (0,1]} \min(\alpha, A_\alpha(x))$ where $\forall x \in A_\alpha$, $A_\alpha(x) = 1$, and 0 otherwise.

The *fuzzy α-cut* (or *fuzzy α-cut*) $A_{\tilde{\alpha}}$ of *A* is defined as $A_{\tilde{\alpha}}(x) = A_{[\tilde{\alpha},1]}(A(x))$, where $[\tilde{\alpha},1]$ $\tilde{\alpha} > 0$ is an interval. A fuzzy *α*-cut can be understood as the set of elements whose membership values are greater than "approximately *α*", *i.e.*, belong to a fuzzy interval $[\tilde{\alpha},1]$.

For example, let a fuzzy set *A* be:

$A = 0.2/7 + 0.4/8 + 0.6/9 + 1.0/10 + 0.6/11 + 0.4/12 + 0.2/13$

Take $\alpha = 0.4$. Then, the *α*-cut A_α of *A* is a crisp set: $A_{0.4} = \{8,9,10,11,12\}$; the strong *α*-cut A_α of *A* is $A_{\bar{.4}} = \{9,10,11\}$; and the fuzzy *α*-cut $A_{\tilde{\alpha}}$ is $A_{\tilde{.4}} = 0.4/8 + 0.6/9 + 1.0/10 + 0.6/11 + 0.4/12$.

2.3.5 Fuzzy relation

2.3.5.1 Fuzzy relation

Let $X_1, X_2, \cdots, X_n$ be *n* universes. An *n-ary fuzzy relation R* on $X_1 \times X_2 \times \cdots \times X_n$ is a fuzzy set in $X_1 \times X_2 \times \cdots \times X_n$. An ordinary relation is a particular case of a fuzzy relation. A fuzzy set *R* on $X \times Y$ is called a *binary fuzzy relation from X to Y*, *i.e.*, there is a mapping $R : X \times Y \to [0,1]$. Its membership function μ_R decides the relation degree of ordered pairs (x, y) . If $\mu_R(x, y)$ takes only value 0 or 1, then *R* is a binary crisp relation. If *X*=*Y*, then *R* is called a *(binary) fuzzy relation on X.*

A binary fuzzy relation is usually represented through a fuzzy matrix. Let $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$. Then the fuzzy relation *R* on $X \times Y$ can be represented by the following *m*n* matrix:

$$R = \begin{bmatrix} r_{11} \; r_{12} \; \cdots r_{1n} \\ r_{21} \; r_{22} \; \cdots r_{2n} \\ \cdots \\ r_{m1} \; r_{m2} \; \cdots r_{mn} \end{bmatrix}$$

where $r_{ij} = \mu_R(x_i, y_j), i = 1,..., m; \ j = 1..., n$.

Let *R* be a fuzzy relation on $X \times X$, then *R* is *reflexive* if $R(x,x) = 1 \quad \forall x \in X$. *R* is *irreflexive* if $R(x,x) = 0 \ \forall x \in X$. Other reflexive properties such as ε-reflexive and weakly reflexive can also be found in the literature. *R* is *symmetric* if $R(x, y) = R(y, x) \ \forall x, y \in X$. *R* is *perfectly antisymmetric* if $\forall x, y \in X$, $x \neq y$, and $R(x, y) > 0$, then $R(y, x) = 0$. *R* is *antisymmetric* if $x \neq y$, then $R(y,x) = R(x, y) = 0$, or $R(y,x) \neq R(x, y)$. *R* is *(max-min) transitive* if $\forall x, y, z \in X$, $R \circ R \subseteq R$, or $R(x,z) \geq \min(R(x, y), R(y,z))$.

A fuzzy relation is called *fuzzy preorder* if it is reflexive and transitive. A fuzzy relation is called a *fuzzy partial ordering* if it is reflexive, transitive and perfectly antisymmetric.

Let *X* be a Cartesian product of universes, $X = X_1 \times X_2 \times \cdots \times X_n$, and $A_1, A_2, \cdots, A_n$ be *n* fuzzy sets in $X_1, X_2, \cdots, X_n$, respectively. The *Cartesian product* of $A_1, A_2, \cdots, A_n$ is defined as $A_1 \times A_2 \times \cdots \times A_n = \sum \min(\mu_{A_1}(x_1), \cdots, \mu_{A_n}(x_n)) / (x_1, \cdots, x_n)$. The Cartesian product of fuzzy sets *A* and *B* is then written as:

$$(A \times B)(x, y) = \min(A(x), B(y)) \quad \text{where} \quad A \subseteq X, B \subseteq Y \ \ x \in X, y \in Y.$$

Obviously, the Cartesian product $A \times B$ is a binary fuzzy relation.

2.3.5.2 Fuzzy mapping

Let $f: X \to Y$ be an ordinary (crisp) mapping from *X* to *Y*. The extension $f^{\to}$ is called a *fuzzy mapping on* $\wp(X)$ *to* $\wp(Y)$ *induced by f* such that (Chang 1968):

$$f^{\to}: \wp(X) \to \wp(Y), f^{\to}(A)(y) = \bigvee \{A(x) : x \in X, f(x) = y\}.$$

$f^{\leftarrow}$ is the inverse image of $f^{\to}$:

$$f^{\leftarrow}: (\wp(Y) \to \wp(X), f^{\leftarrow}(B)(x) = B(f(x));$$

where $A \in \wp(X), B \in \wp(Y)$, $x \in X, y \in Y$.

2.3.5.3 Fuzzy composition

Let $C \in \wp(X)$ be a fuzzy set, and $R \in \wp(X \times Y)$ be a fuzzy relation from *X* to *Y*. The *sup-min composition* is defined as:

$$C \circ R(y) = \sup_{x \in X} \min(C(x), R(x, y)) \quad \forall y \in Y.$$

Take an example. Let *C* be a fuzzy set in the universe $\{1,2,3\}$, and *R* be a binary relation on $\{1,2,3\}$. Assume that $C = 0.2/1 + 1/2 + 0.2/3$ and

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{bmatrix} \end{array}$$

Then the sup-min composition will be

$$C \circ R = [0.2/1 + 1/2 + 0.2/3] \circ \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{bmatrix} = 0.8/1 + 1/2 + 0.8/3$$

Let R be a fuzzy relation on $X \times Y$, and Q be a fuzzy relation on $Y \times Z$. The *sup-min composition* of R and S is defined as $R \circ S$, where $R \circ S(x,z) = \sup_{y \in Y} \min(R(x,y), S(y,z)) \quad \forall x \in X, \forall z \in Z$. For example, let

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0.7 \\ 0 & 0.8 & 0 & 0 \\ 0.9 & 1 & 0.7 & 0.8 \end{bmatrix} \end{array}, \quad S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.9 & 0.3 \\ 0 & 0.4 & 0 \\ 0.9 & 0.5 & 0.8 \\ 0.6 & 0.7 & 0.5 \end{bmatrix} \end{array},$$

then

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.8 & 0.5 \\ 0 & 0.4 & 0 \\ 0.7 & 0.9 & 0.7 \end{bmatrix} \end{array}$$

2.4 Fuzzy reasoning

2.4.1 Fuzzy logic

In Boolean logic, a *proposition* is a statement that is either true (1) or false (0). {0,1} is called the *truth value*. For example, "*Peter is a man*" is a proposition. A *fuzzy proposition* is a statement whose truth value is in [0,1]. The general form of a fuzzy proposition is "*A: h is F*", where *h* is a *variable* and *F* is a fuzzy set. For example, "*Peter is a tall man*" is a fuzzy proposition. The logic that studies fuzzy propositions is called *fuzzy logic* (Zadeh 1988). The operations between fuzzy propositions such as *and*, *or*, *not* can be defined as the operations defined for two fuzzy sets.

2.4.2 Fuzzy implication

In Boolean logic, let *p* and *q* be crisp propositions. The *implication* $p \Rightarrow q$ can be defined by a *truth table* (Table 2.1).

Table 2.1 Truth table of implication

p	q	$p \Rightarrow q$
1	1	1
0	1	1
0	0	1
1	0	0

Let *A*, *B* be fuzzy propositions. A *fuzzy implication* is a fuzzy relation from *A* to $B: (A \Rightarrow B)(x, y) = A(x) \Rightarrow B(y)$. The truth value of $A \Rightarrow B$ is decided by fuzzy sets $A(x), B(y)$. Take an example, let *A* and *B* be fuzzy propositions: *A: the pressure is high, B: the volume is small.* Represent a fuzzy proposition $A = 0/1 + 0.25/2 + 0.5/3 + 0.75/4 + 1/5$, where $\{1,2,3,4,5\}$ denotes the pressure value, and $\{0,0.25,0.5,0.75,1\}$ is the membership degree of the high pressure. $B = 1/1 + 0.8/2 + 0.5/3 + 0.2/4 + 5/5$. Assume the pressure is 4 and the volume is 1. Then a fuzzy implication $A \Rightarrow B$ will be represented by:

$$(A \Rightarrow B)(4,1) = A(4) \Rightarrow B(1) = .75 \Rightarrow 1$$

Denote $u \in U$ as a variable of *A*, and *U* is the set of variables. Denote $v \in V$ as a variable of *B*. $A(u), B(v)$ represents the truth values of propositions *A*, *B*. The truth value of a fuzzy implication can be defined in several ways:

(1) Larsen: $A(u) \Rightarrow B(v) = A(u) \cdot B(v)$;

(2) Lukasiewicz: $A(u) \Rightarrow B(v) = \min(1, 1 - A(u) + B(v))$;

(3) Mamdani: $A(u) \Rightarrow B(v) = \min(A(u), B(v))$;

(4) Standard strict: $A(u) \Rightarrow B(v) = \begin{cases} 1 \ \ if \ \ A(u) \le B(v)) \\ 0 \ otherwise \end{cases}$;

(5) Gödel: $A(u) \Rightarrow B(v) = \begin{cases} 1 \ \ if \ \ A(u) \le B(v)) \\ B(v) / A(u) \ otherwise \end{cases}$;

(6) Kleene-Dienes: $A(u) \Rightarrow B(v) = \max(1 - A(u), B(v))$;

(7) Kleene-Dienes-Lûk: $A(u) \Rightarrow B(v) = 1 - A(u) + A(u) \cdot B(v)$.

In the above example, if the Mamdani operator is adopted, then the fuzzy implication is:

$$(A \Rightarrow B)(4,1) = A(4) \Rightarrow B(1) = 0.75 \Rightarrow 1 = \min(0.75, 1) = 0.75 .$$

Let *A* be a fuzzy set of $\wp(X)$. A *modifier* is a fuzzy set generated from *A* and $\lambda > 0$ such that $H_\lambda : \wp(X) \to \wp(X) = \{x \in X : (A(x))^\lambda\}$. For example, we can define the fuzzy sets "*very young*", or "*more or less young*" from fuzzy set "*young*" by

very $A(x) = (A(x))^2$, *more or less* $A(x) = \sqrt{A(x)}$, where *A* is a fuzzy set *young*.

2.4.3 Principles of fuzzy reasoning

In 1975, Zadeh introduced the theory of approximate reasoning. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the presentation of propositions as statements assigning fuzzy sets as values to variables and the composition rules of inference.

Suppose there are two interactive variables $x \in X$, $y \in Y$ (*X*, *Y* are universes), and the causal relationship between *x* and *y* is completely known, *i.e.*, we know that *y* is a function of *x*: $y = f(x)$. Then we can make inferences easily by:

$$\begin{array}{l} \text{Premise}: \; y = f(x) \\ \text{Fact}: x = x' \\ \hline \text{Consequence}: \; y' = f(x') \end{array}$$

The above *inference rule* says that if we have $y = f(x)$, and now $x = x'$, then *y* takes the value $y' = f(x')$.

The basic problem of approximate reasoning is to find the membership function of the consequence *B* from the *rule base* $\{R_1, R_2, ..., R_n\}$:

$$\begin{array}{l} R_1 : if \;\; x \; is \; A_1, \;\; then \; y \; is \; B_1 \\ R_2 : if \;\; x \; is \; A_2, \;\; then \; y \; is \; B_2 \\ ... \\ R_n : if \;\; x \; is \; A_n, \;\; then \; y \; is \; B_n \\ \text{Fact}: x \; is \; A \\ \hline \text{Consequence}: \; y \; is \; B \end{array}$$

In order to reach the final decision, several translation rules are defined:

Entailment rule:

$$\begin{array}{l} if \;\; x \; is \; A \\ A \subseteq B \\ \hline x \; is \; B \end{array}$$

Conjunction rule:

$$\begin{array}{l} if \;\; x \; is \; A \\ if \;\; x \; is \; B \\ \hline x \; is \; A \cap B \end{array}$$

Disjunction rule:

$$\begin{array}{l} if \;\; x \; is \; A \; or \\ if \;\; x \; is \; B \\ \hline x \; is \; A \cup B \end{array}$$

Projection rule:

$$\frac{\textit{if } (x,y) \textit{ have relation } R}{x \textit{ is } P_X(R)} \quad \text{(Example)} \quad \frac{\textit{if } (x,y) \textit{ is close to } (3,2)}{x \textit{ is close to } 3}$$

Negation rule:

$$\frac{\textit{if } not(x \textit{ is } A)}{x \textit{ is } \neg A}$$

The most important rule in approximate reasoning is the *compositional rule of inference*. Let:

$$\begin{array}{l} R: \textit{if } x \textit{ is } A, \textit{then } y \textit{ is } B \\ Fact: x \textit{ is } A' \\ \hline \text{Consequence:} \quad y \textit{ is } B' \end{array}$$

where the consequence *B'* is determined as the composition of fact and the fuzzy implication operator:

$$B' = A' \circ (A \Rightarrow B)$$

In approximate reasoning, the sup-min composition is the usual compositional method. The fuzzy implication may vary.

If we take the above Lukasiewicz implication: $A(u) \Rightarrow B(v) = \min(1, 1 - A(u) + B(v))$. Then

$$B'(v) = A' \circ (A \Rightarrow B) = \sup_{u \in U}(\min(A'(u), \min(1, 1 - A(u) + B(v)))) \ \ v \in V$$

For the Mamdani implication: $A(u) \Rightarrow B(v) = \min(A(u), B(v))$. The composition rule will be:

$$B'(v) = A' \circ (A \Rightarrow B) = \sup_{u \in U}(\min(A'(u), \min(A(u), B(v)))) \ \ v \in V .$$

2.4.4 Process of fuzzy reasoning

Let $\{R_1, R_2, ..., R_n\}$ be a rule base, and x_0 be a crisp value of input *X*.

$$\begin{array}{l} R_1: \textit{if } x \textit{ is } A_1, \textit{then } z \textit{ is } C_1 \\ R_3: \textit{if } x \textit{ is } A_2, \textit{then } z \textit{ is } C_2 \\ \ldots \\ R_n: \textit{if } x \textit{ is } A_n, \textit{then } z \textit{ is } C_n \\ Fact: x \textit{ is } x_0 \\ \hline \text{Consequence:} \quad z \textit{ is } C \end{array}$$

The approximate reasoning is to find *C* when the input is x_0. The basic inference process is:

(1) Input x_0;

(2) Fuzzifying x_0 in each fuzzy proposition to get the fire strength of each rule: $A_i(x_0), i = 1...n$;

(3) Calculating the output of each rule by $C_i^{'}(z) = A_i(x_0) \Rightarrow C_i(z), i = 1...n$;

(4) Calculating the overall output by $C(z) = C_1 \cup C_2 \cup ... \cup C_n = C_1^{'}(z) \cup ... \cup C_n^{'}(z)$.

We illustrate the fuzzy reasoning process with Mamdani's implication in Figure 2.7. Suppose the rule number n=2.

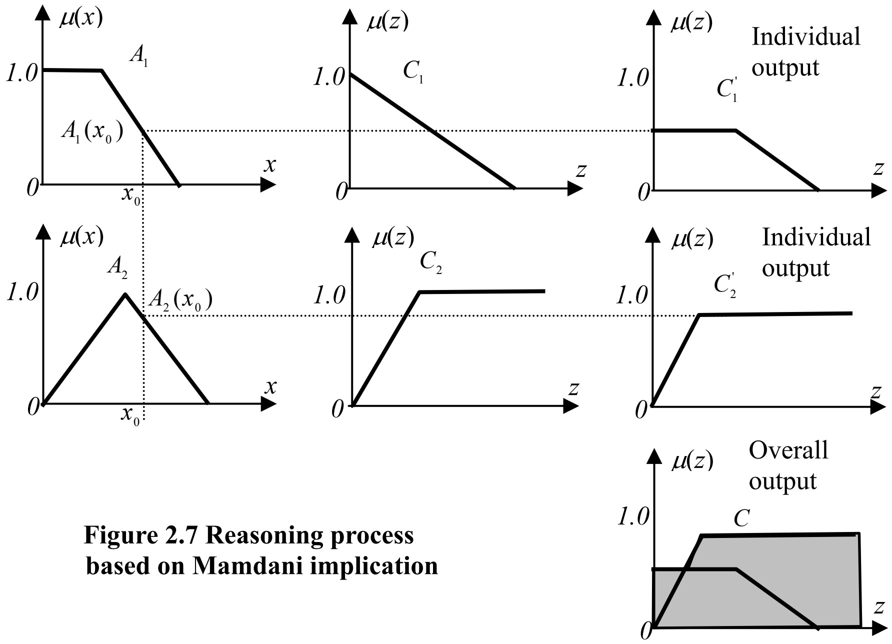


Figure 2.7 Reasoning process based on Mamdani implication

The output in the process is a fuzzy set. In many applications, the fuzzy set should be transformed into a crisp value by a *defuzzifier*:

$$z_0 = defuzzufier(C)$$

The *defuzzification* is a process to select a representative element from the fuzzy output *C*. The most defuzzification operators are *center-of-area* and *mean-of-max* (Figure 2.8).

(1) Center of area

$$z_0 = \frac{\sum_{j=1}^{n} z_j C(z_j)}{\sum_{j=1}^{n} C(z_j)}$$

(2) Mean of max:

$$z_0 = \frac{1}{n}\sum_{j=1}^{n} z_j$$

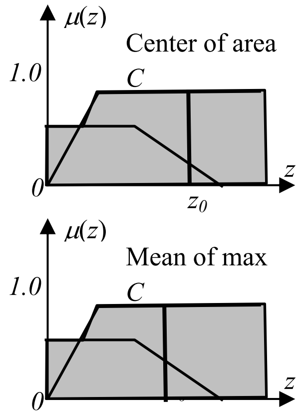


Figure 2.8 Center of area and mean of max

2.4.5 Fuzzy reasoning methods

In general, the task of fuzzy reasoning is to get the output from the input based on the rule base:

$R_1 : if \;\; x \; is \; A_1 \; and \; y \; is \; B_1, then \; z \; is \; C_1$

$R_2 : if \; x \; is \; A_2 \; and \; y \; is \; B_2, then \; z \; is \; C_2$

...

$R_n : if \; x \; is \; A_n \; and \; y \; is \; B_n, then \; z \; is \; C_n$

$Fact : x \; is \; x_0 \; and \; y \; is \; y_0$

Consequence: *y is C*

There are many methods for deriving the output *C*. In the following, the Mamdani method and the Tsukamoto method are described. For simplicity, let *n*=2.

A. The Mamdani method (Mamdani 1976):

(1) The firing levels of each rule, denoted by $\alpha_i, i = 1,2$; are computed by $\alpha_1 = \min(A_1(x_0), B_1(y_0))$ and $\alpha_2 = \min(A_2(x_0), B_2(y_0))$;

(2) The individual rule output is $C_1'(z) = \min(\alpha_1, C_1(z))$ and $C_2'(z) = \min(\alpha_2, C_2(z))$;

(3) The overall output is computed by the sup-min composition: $C(z) = C_1'(z) \vee C_2'(z) = \min(\alpha_1, C_1(z)) \vee \min(\alpha_2, C_2(z))$.

The process can be illustrated by Figure 2.9:

The crisp *z* can be calculated by using a defuzzifier.

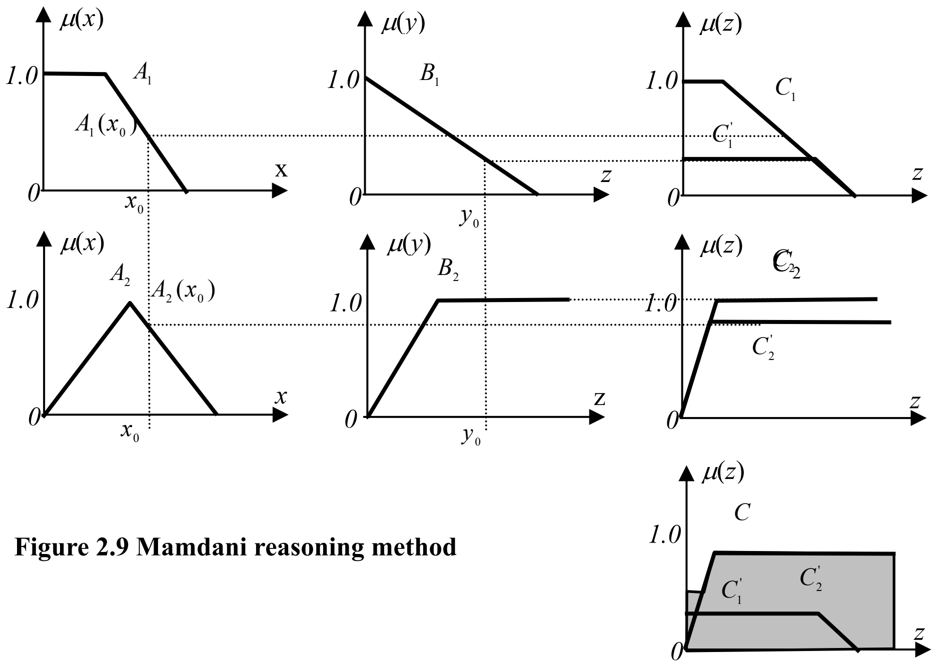


Figure 2.9 Mamdani reasoning method

B: The Tsukamoto method:

(1) The fire level of each rule is calculated by $\alpha_1 = \min(A_1(x_0), B_1(y_0))$ and $\alpha_2 = \min(A_2(x_0), B_2(y_0))$;

(2) The individual crisp outputs of z_1 and z_2 are computed from the equations $\alpha_1 = C_1(z_1), \alpha_2 = C_2(z_2)$;

(3) The overall crisp action is calculated by the center of area and is expressed as $z_0 = \dfrac{\alpha_1 z_1 + \alpha_2 z_2}{\alpha_1 + \alpha_2}$

2.5 Fuzzy topology

2.5.1 Fuzzy topological space

Fuzzy topology is constructed based on fuzzy (sub)sets. It is an extension of general (crisp) topology and has several definitions. The notion introduced here is based on the definition proposed by Chang (1968).

Let *A* be a fuzzy subset of an ordinary (crisp) set *X*, and $\wp(X)$ be the fuzzy power set of *X*. $\forall \delta \subseteq \wp(X)$ if

$$\varnothing, X \subseteq \delta$$

$$\forall A_i \in \delta, \bigcup_i A_i \in \delta$$

$$\forall U, V \in \delta, U \cap V \in \delta$$

Then δ is called a *fuzzy topology on X* ($i \in I$ is an index set). (X, δ) is called a *fuzzy topological space* (or *fts* for short). When there is no confusion, we write simply X instead of (X, δ) and write crisp (general) topological space as *cts* for short. Every element of δ is called an *open (fuzzy) set* in (X, δ). A set A is a *closed (fuzzy) set* if its complement A^c is open. The union of all open sets contained in A is the *interior* of A, denoted by A^o. A^o is the largest open set contained in A. The intersection of all the closed sets containing A is called the *closure* of A, denoted by A^-. A^- is the smallest closed set containing A. The *exterior* of A is the complement of A^- and is denoted by A^e. Obviously A^e is an open set. An open set A is called *regular open* if it is equal to the interior of the closure of A. A closed set A is called *regular closed* if it is equal to the closure of the interior of A.

A fuzzy set in X is called *a fuzzy point* iff it has the membership degree 0 for all $y \in X$ except one, say $x \in X$ (Wong 1974). We denote a fuzzy point by x_λ $(0 < \lambda \le 1)$, *i.e.*, the value at x is λ, and call the point x its *support*. λ is the height of x_λ. The fuzzy point x_λ is contained in a fuzzy set A or belongs to A, denoted by $x_\lambda \in A$ iff $\lambda \le A(x)$. $A(x)$ denotes the membership degree of x to A.

Let (X, δ_1), (X, δ_2) be two fuzzy topological spaces. (X, δ_1) is said to be *finer* than (X, δ_2), or (X, δ_2) is said to be *coarser* than (X, δ_1) if $\delta_1 \supset \delta_2$.

Let Y be a crisp subset of fts (X, δ). Y is called the *subspace* of (X, δ) if a fuzzy subset A of Y is open in Y if and only if $A = X \cap U$ for some open subset U of X.

In general the definitions of fts and cts are the same in their forms. The main difference between fts and cts is that an fts contains fuzzy sets but a cts contains only ordinary (crisp) subsets. The other differences between fts and cts are caused by the differences between the properties of fuzzy sets and crisp sets.

Between fuzzy sets A and B, Propositions 2.1, 2.2 and 2.3 on crisp sets A and B still hold (Liu and Luo 1997):

Proposition 2.7 *Let A, B be two fuzzy sets in fts X.*

(1) $A^o \subseteq A$, $(A^o)^o = A^o$;

(2) $A \subseteq B \Rightarrow A^o \subseteq B^o$;

(3) $(A \cap B)^o = A^o \cap B^o$, $(A \cup B)^o \supseteq A^o \cup B^o$.

According to Proposition 2.7(3), the intersection of two regular open sets is still regular open.

Proposition 2.8 *Let A, B be two fuzzy sets in fts X.*

(1) $A \subseteq A^-, (A^-)^- = A^-$;

(2) $A \subseteq B \Rightarrow A^{-} \subseteq B^{-}$;

(3) $(A \cap B)^{-} \subseteq A^{-} \cap B^{-}$, $(A \cup B)^{-} = A^{-} \cup B^{-}$.

According to Proposition 2.8(3), the union of two regular closed sets is still regular closed.

Proposition 2.9 *Let A be a fuzzy set in fts X.* $A^{oc} = A^{c-}, A^{-c} = A^{co}$.

However, Proposition 2.4 does not hold in general. For example, let $X = \{a, b, c, d\}$, $\delta = \{X, \varnothing, \{a_{0.2}\}, \{a_{0.2}, b_{0.2}\}\}$. Then δ is a fuzzy topology on *X*. Suppose $A = \{a_{0.3}\}$, *i.e.*, *A* is a set of a fuzzy point. Then $A^{o} = \{a_{0.2}\}$; $A^{oc} = X - A^{o} = A^{oc} = \{a_{0.8}, b, c\}$; $A^{c} = X - A = \{a_{0.7}, b, c\}$; $A^{c-} = \{a_{0.8}, b, c\}$; $A^{-} = \{a_{0.3}\}^{-} = \{a_{0.8}, b_{0.8}, c\}$; $A^{-c} = \{a_{0.2}, b_{0.2}\}$. Obviously, $A^{oc} = A^{c-}, A^{-c} = A^{co}$. This example shows Proposition 2.9 holds. But $A \cup A^{c} = \{a_{0.3}\} \cup \{a_{0.7}, b, c\} = \{a_{0.7}, b, c\} \neq X$.

Furthermore, $A^{o} \cup A^{oc} = \{a_{0.2}\} \cup \{a_{0.8}, b, c\} = \{a_{0.8}, b, c\} \neq X$ and
$A^{-} \cup A^{-c} = \{a_{0.8}, b_{0.8}, c\} \cup \{a_{0.2}, b_{0.2}\} = \{a_{0.8}, b_{0.8}, c\} \neq X$

This problem is caused by the excluded-middle law since it does not hold in fuzzy set theory. The properties of the fuzzy boundary of a fuzzy set will be discussed in Chapter 3.

2.5.2 Neighborhood and quasi-neighborhood

A fuzzy set *A* in (X, δ) is called *a neighborhood* of fuzzy point x_{λ} if there exists a $B \in \delta$ such that $x_{\lambda} \in B$ and $B \subseteq A$ (Pu and Liu 1980). A fuzzy point $x_{\lambda} \in A^{o}$ iff x_{λ} has a neighborhood contained in *A*. Obviously, a fuzzy point $x_{\lambda} \notin A^{o}$ iff every neighborhood of x_{λ} is not contained in *A*. A fuzzy point x_{λ} is called *quasi-coincident* with *A*, denoted by $x_{\lambda} \hat{q} A$, iff $\lambda > A^{c}(x)$ or $\lambda + A(x) > 1$. Call fuzzy set *A* *quasi-coincident* with *B* if $A(x) > B^{c}(x)$ or $A(x) + B(x) > 1$, $\forall x \in X$. A fuzzy set *A* in (X, δ) is called a *quasi-neighborhood* of x_{λ} if there exists a $B \in \delta$ such that $x_{\lambda} \hat{q} B$ and $B \subseteq A$. A fuzzy point $x_{\lambda} \in A^{-}$ iff each quasi-neighborhood of x_{λ} is quasi-coincident with *A*.

Compared with the neighborhood of a crisp topological space, it can be perceived that a fuzzy topological space possesses more neighborhood structures. For further discussion we define *pan-neighborhood P* of fuzzy point x_{λ} if there exists a $B \in \delta$ such that $B \subseteq P$ and $x_{\lambda} \cap B \neq \varnothing$ (Tang and Kainz 2002). Obviously $A(x) = A^{-}(x)$ iff there is a pan-neighborhood $P \in \delta$ of *x* such that $A(x) + P(x) = 1$; $A(x) = A^{o}(x)$ iff there is a *P* of *x* such that $A(x) = P(x)$.

These neighborhood concepts are illustrated in Figure 2.10. Let *X* be a closed interval

$[c_1, c_2]$ of *R*; *U* is defined as an open set of *X*. $\{\varnothing, U, X\}$ is a fuzzy topology on *X*. *U* is a neighborhood of fuzzy point x_a, U^c is a quasi-neighborhood of fuzzy point y_b, *U* is a pan-neighborhood of z_c.

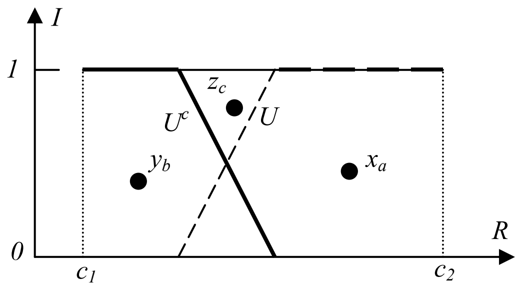


Figure 2.10 Neighborhood, quasi-neighborhood and pan-neighborhood

2.5.3 Separation and connectedness

Two fuzzy sets *A* and *B* in (X, δ) are said to be *separated* if there exist $U, V \in \delta$ such that $U \supseteq A, V \supseteq B$ and $U \cap B = V \cap A = \varnothing$. Two fuzzy sets *A* and *B* in (X, δ) are said to be *Q-separated* if there exist closed sets *H*, *K* such that $H \supseteq A, K \supseteq B$ and $H \cap B = K \cap A = \varnothing$ (Pu and Liu 1980).

In general, Q-separation and separation do not imply each other. However, we have:

Proposition 2.10 *Let A and B be two crisp sets in an fts, then A and B are separated iff they are Q-separated.*

A fuzzy topological space *X* is called *connected* if there is no separated *C* and *D*, such that $X = C \cup D$ (Liu and Luo 1997). We define the connectedness of a fuzzy set by separation and Q-separation. A fuzzy set *A* is said to be *open-connected* if there is no separated *C* and *D*, such that $A = C \cup D$. A fuzzy set *A* is said to be *closed-connected* if there is no Q-separated *C* and *D*, such that $A = C \cup D$. A fuzzy set is said to be *double-connected* if it is both open-connected and closed-connected.

2.5.4 Fuzzy homeomorphism and fuzzy topological relation

Let $f : X \to Y$ be an ordinary mapping between *X* and *Y*, and (X, δ), (Y, σ) be fts. The *fuzzy mapping* from fts (X, δ) to fts (Y, σ) and its *inverse mapping* are defined as (Chang 1968):

$$f^{\to} : (X, \delta) \to (Y, \sigma), f^{\to}(A)(y) = \bigvee\{A(x) : x \in X, f(x) = y\}, A \in \wp(X), y \in Y$$
$$f^{\leftarrow} : (Y, \sigma) \to (X, \delta), f^{\leftarrow}(B)(x) = B(f(x)), B \in \wp(Y), x \in X$$

Call $f^{\to} : (X, \delta) \to (Y, \sigma)$ *(fuzzy) continuous* if its inverse mapping

$f^{\leftarrow}:(Y,\sigma)\to(X,\delta)$ maps every open subset in (Y,σ) to an open subset in (X,δ). Call it *open* if it maps every open subset in (X,δ) to an open subset in (Y,σ). $f^{\rightarrow}:(X,\delta)\to(Y,\sigma)$ is called *(fuzzy) homeomorphism* if it is bijective, continuous and open. Fuzzy homeomorphisms are *union preserving* and *crisp subset preserving* (Liu and Luo 1997). The properties of a fuzzy set that are invariant under fuzzy homeomorphisms are *(fuzzy) topological properties* (or *topological invariants*).

Let *R* be a binary fuzzy relation *R* from fuzzy set $A\subseteq X$ to fuzzy set $B\subseteq X$ on fuzzy topological space *X*. *R* is called a *fuzzy topological relation from A to B on X* if *R* keeps topological invariants under a fuzzy homeomorphism. If $\mu_R(a,b)$ $(a\in A,b\in B)$ takes only value 0 and 1, then *R* is crisp, otherwise it is fuzzy.

2.5.5 T_0, T_1, T_2, regular and normal space

The following notions are summarized from Liu and Luo (1997). Let (X,δ) be a fuzzy topological space. (X,δ) is called *quasi-T_0* if for every two distinguished points x_λ, x_σ with the same support *x*, there exists a quasi-neighborhood *A*, such that $x_\lambda \,\bar{\hat{q}}\, A^c(x)$ (*i.e.*, $\lambda\le A^c(x)$ or $\lambda+A(x)\le 1$), or a quasi-neighborhood *B*, such that $x_\sigma \,\bar{\hat{q}}\, B^c(x)$ (*i.e.*, $\sigma\le B^c(x)$ or $\sigma+B(x)\le 1$). (X,δ) is called *sub-T_0* if for two distinguished ordinary points x, y, there exists a quasi-neighborhood *A* and $\lambda\in(0,1]$, such that $x_\lambda \,\bar{\hat{q}}\, A^c(x)$, or a quasi-neighborhood *B*, such that $x_\lambda \,\bar{\hat{q}}\, B^c(x)$. (X,δ) is called T_0 if for every two distinguished points x_λ, y_σ, there exists a quasi-neighborhood *A*, such that $x_\lambda \,\bar{\hat{q}}\, A^c(x)$, or a quasi-neighborhood *B*, such that $x_\sigma \,\bar{\hat{q}}\, B^c(x)$. Every T_0 space is both *quasi-T_0* and *sub-T_0*.

Let (X,δ) be an fts. (X,δ) is called T_1 if for every two distinguished points x_λ, y_σ, there exists a quasi-neighborhood *A* of x_λ such that $y_\sigma \,\bar{\hat{q}}\, A$. Obviously, every T_1 space is T_0.

Let (X,δ) be an fts. (X,δ) is called T_2 if, for every two distinguished points $x_\lambda, y_\sigma (x\ne y)$, there exist quasi-neighborhoods *A, B* such that $A\cap B=\varnothing$. In general a T_2 space cannot imply T_0 or even quasi-T_0.

Let (X,δ) be an fts. (X,δ) is called *regular* (or T_3) if for every open set *U*, there exists $\mathrm{A}\subseteq\delta$, such that $\bigcup\mathrm{A}=U$ and $A^-\subseteq U$ for every $A\in\mathrm{A}$. (X,δ) is called *p-regular* (or *p-T_3*) if, for every closed set *V* and every point x_λ, $x\notin \mathrm{supp}(V)$, there exists quasi-neighborhood *A* of x_λ and *B* of *V* such that $U\cap V=\varnothing$. T_3 and *p-T_3*

do not imply each other. Every p-T_3 space is T_2.

Let (X,δ) be an fts. (X,δ) is called *normal* (or T_4) if, for every closed set C and every open set A such that $C \subseteq A$, there exists an open set B such that $C \subseteq B \subseteq A$. (X,δ) is called *p-normal* (or *p*-T_4) if, for every two closed subsets A and B such that $\text{supp}(A) = \text{supp}(B) = \varnothing$, there exist quasi-neighborhood U of A and quasi-neighborhood V of B such that $U \cap V = \varnothing$.

The relationships between these fuzzy topological spaces (T_i, i=0,1,2,3,4) are illustrated in Figure 2.11.

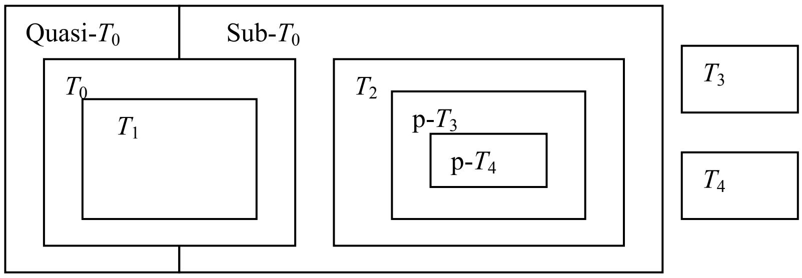


Figure 2.11 Relationships between fuzzy topological spaces

2.5.6 Induced space

In Section 2.5.4 we have seen that a fuzzy topological space has more complicated structures than a crisp topological space. In order to show the relations between fuzzy topological space and crisp topological space, we introduce three fuzzy topological spaces: the stratified space, the weakly induced space and the induced space (Liu and Luo 1997).

Let (X,δ) be an fts. δ is called *stratified* if δ contains every layer of X, *i.e.*, for every $\alpha \in [0,1]$, $X_\alpha \in \delta$. (X,δ) is called a *stratified topological space* if δ is stratified. Lowen (1976) defined a fuzzy topology on X, replacing $\varnothing \in \delta, X \in \delta$ by every level $X_\alpha \in \delta$.

Let (X,δ) be an fts. For every $A \in \delta$, supp(A) is a crisp set. Let $[\delta] = \{\text{supp}(A) : A \in \delta\}$, then $(X,[\delta])$ forms a crisp topological space.

Let [0,1] be the unit interval. The crisp topology that is generated by subsets $A = \{[0,a) : 0 \le a \le 1\}$ is called *upper topology on* $[0,1]$. The crisp topology that is generated by subsets $A = \{(a,1] : 0 \le a \le 1\}$ is called *lower topology on* $[0,1]$. The topology that is generated by subsets $A = \{(a,b) : 0 \le a < b \le 1\}$ is called *interval topology on* $[0,1]$. Let (X,δ) be a crisp topological space. A mapping

$f:X\to[0,1]$ is called *upper semicontinuous* if *f* is continuous for the upper topology on $[0,1]$. A mapping $f:X\to[0,1]$ is called *lower semicontinuous* if *f* is continuous for the lower topology on $[0,1]$. A mapping $f:X\to[0,1]$ is continuous iff *f* is both upper semicontinuous and lower semicontinuous.

Let (X,δ) be an fts. δ is called *weakly induced fuzzy topology on X* if every $U\in\delta$ is a lower semicontinuous mapping from $(X,[\delta])$ to $[0,1]$. δ is called *induced fuzzy topology on X* if δ is exactly the family of all the lower semicontinuous mappings from $(X,[\delta])$ to $[0,1]$.

Between these topological spaces, the following propositions hold:

Proposition 2.11 *Let (X,δ) be an fts. Then (X,δ) is induced iff (X,δ) is both stratified and weakly induced.*

Proposition 2.12 *Let (X,δ) be an fts, A be a fuzzy set of X, A_α be a α-cut of fuzzy set A, and $(A_\alpha)^o$ be the open set of A_α in $(X,[\delta])$.*

(1) (X,δ) *is stratified iff* $\delta\supseteq\{\bigvee_{0<\alpha\le1}(\alpha\wedge(A_\alpha)^o(x)):A\in\wp(X)\}$;

(2) (X,δ) *is weakly induced iff* $\delta\subseteq\{\bigvee_{0<\alpha\le1}(\alpha\wedge(A_\alpha)^o(x)):A\in\wp(X)\}$;

(3) (X,δ) *is induced iff* $\delta=\{\bigvee_{0<\alpha\le1}(\alpha\wedge(A_\alpha)^o(x)):A\in\wp(X)\}$.

Proposition 2.13 *Let (X,δ) be an induced fts, A be a fuzzy set of (X,δ), A_α be an α-cut of fuzzy set A, and $(A_\alpha)^o$ be the open set of A_α in $(X,[\delta])$.*

(1) $A^-=\bigvee_{0<\alpha\le1}(\alpha\wedge(A_\alpha)^-(x))$;

(2) $A^o=\bigvee_{0<\alpha\le1}(\alpha\wedge(A_\alpha)^o(x))$.

Proposition 2.14 *Let (X,δ) be an induced fts. (X,δ) is connected iff $(X,[\delta])$ is connected.*

We use an example to show these spaces. Let (R,d) be a crisp Euclidean space, and $D^o=(x,y)$ ($x<y$) be an open interval of *R*. Let $\widetilde{R}$ be the extension of *R* with fuzzy points x_a $(0<a\le1)$. Define a *fuzzy open interval D* between x_a,y_b $(x<y,0<a,b\le1)$ in $\widetilde{R}$: $D^o=(x_a,y_b)=\{z_c:x<z<y \text{ and } 0<c\le1\}$. The distance between x_a, y_b is defined as $|\text{supp}(x_a)-\text{supp}(y_b)|$. Let us construct an fts $(\widetilde{R},\gamma)$ where $\gamma=\{\widetilde{\rho}:\widetilde{\rho}\subseteq D^o\}$, *i.e.*, the elements of γ are fuzzy open intervals. $(\widetilde{R},\gamma)$ is the stratified space of (R,d) since each level of *R* is contained in $(\widetilde{R},\gamma)$. However, it is not weakly induced. In (R,d), every closed interval $D=[x,y]$ is closed and not open. Every open interval $D^o=(x,y)$ is open but not closed. In

$(\widetilde{R},\gamma)$, every fuzzy closed interval $D=[x_a,y_b], 0<a,b\le 1$ is closed. According to the definition of fuzzy topology, the arbitrary intersection of closed sets is still closed. Therefore, $\lim_{a,b\to 0}(\bigcap D) = \lim_{a,b\to 0}(\bigcap[x_a,y_b]) = (x_a,y_b)$ is also closed (Figure 2.12). Let $c=\min(a,b)$. Then the support of the closed set (x_a,y_b) will be $(x_a,y_b)_c=(x,y)$, which is an open interval in (R,d). However, since (x,y) cannot be closed in (R,d). Therefore, $(\widetilde{R},\gamma)$ is not weakly induced space of (R,d). $(\widetilde{R},\gamma)$ is not connected since it can be the union of two closed intervals.

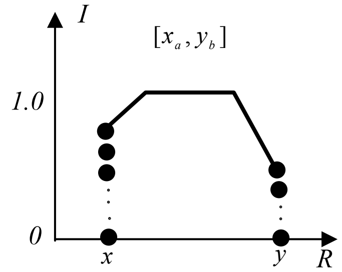


Figure 2.12 Intersection of fuzzy closed intervals

2.5.7 Relationships between crisp spaces and induced spaces

We show the conclusions between stratified, weakly induced, induced topological spaces and crisp topological spaces with consideration of T_i spaces.

Proposition 2.15

(1) *Every stratified space is quasi-T_0. If* $(X,[\delta])$ is T_0, then (X,δ) *is sub-T_0. If* (X,δ) *is weakly induced and sub-T_0, then* $(X,[\delta])$ *is* T_0. *Every induced space* (X,δ) *is* T_0 *iff* $(X,[\delta])$ *is* T_0.

(2) *If* (X,δ) *is weakly induced and* T_1, *then* $(X,[\delta])$ *is* T_1. *Let* (X,δ) *be a stratified space. If* $(X,[\delta])$ *is* T_1 *then* (X,δ) *is* T_1. *Every induced space* (X,δ) *is* T_1 *iff* $(X,[\delta])$ *is* T_1.

(3) *Let* (X,δ) *be a weakly induced topological space. Then* (X,δ) *is* T_2 *iff* $(X,[\delta])$ *is* T_2. *Let* (X,δ) *be an induced space. Then* (X,δ) *is* T_2 *and* T_1 *iff* $(X,[\delta])$ *is* T_2.

(4) *Let* (X,δ) *be a weakly induced space. If* (X,δ) *is p-regular, then* $(X,[\delta])$ *is regular. Every induced space* (X,δ) *is regular or p-regular iff* $(X,[\delta])$ *is regular.*

(5) *If* (X,δ) *is weakly induced and* $(X,[\delta])$ *is normal, then* (X,δ) *is p-normal. If* (X,δ) *is weakly induced and normal, then* $(X,[\delta])$ *is normal. Let* (X,δ) *be an induced space. Then* (X,δ) *is normal or p-normal iff* $(X,[\delta])$ *is normal.*

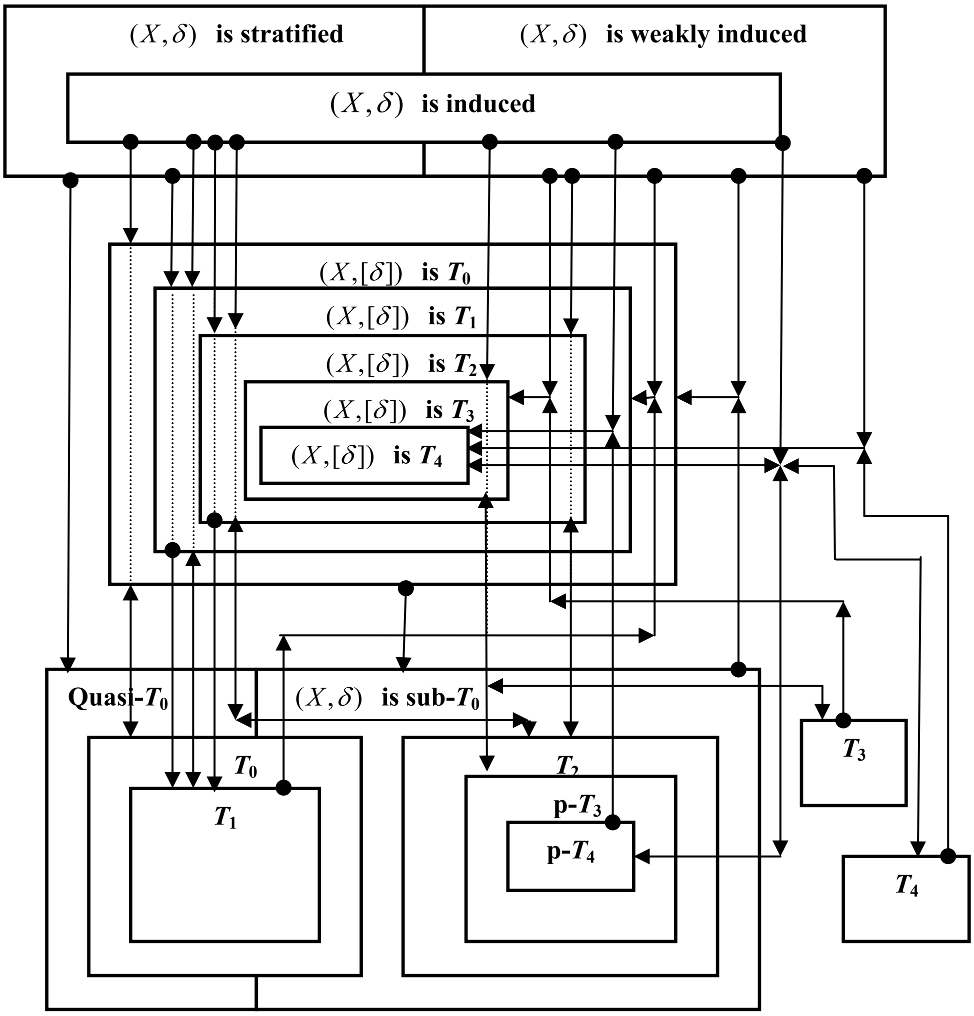


Figure 2.13 Relationships between difference topological spaces

The relations between crisp spaces and fuzzy stratified, weakly induced and induced spaces are illustrated in Figure 2.12, ***A*●——►*B*** stands for *A* implies *B*. ***A* ◄——►*B*** stands for two spaces imply each other.

2.5.8 Fuzzy pseudo-metric space

A mapping $P:\wp(X)\times\wp(X)\to[0,\infty)$ is called *pseudo-metri*c *on* X if $\forall x_a, y_b, u_c, v_d \in X$, P satisfies the following conditions:

(1) If $x_a \geq y_b$, then $P(x_a, y_b) = 0$;

(2) $P(x_a, y_b) \le P(x_a, u_c) + P(u_c, y_b)$;

(3) $P(x_a, y_d) = \bigvee_{y_c \le y_b} \bigwedge_{x_d \le x_a} P(x_d, y_c)$;

(4) $\exists u_c$, $x_c \hat{q} u_c$, $P(y_b, u_c) < r \Leftrightarrow \exists v_d$, $y_b \hat{q} v_d$, $P(x_a, v_d) < r$.

Call (X, P) a *fuzzy pseudo-metric space*. $\forall A \in \wp(X)$, defines a mapping $\hat{f}_r : \wp(X) \to \wp(X)$ for every $r > 0$:

$$\hat{f}_r(A) = \bigvee\{y_b : \exists z_c \in A, P(z_c, y_b) < r\}.$$

Call $D_P = \{\hat{f}_r : r > 0\}$ the *associated neighborhood mappings of P*. The topology generated by *P* is denoted by $\delta(P)$. If (X, P) is a fuzzy pseudo-metric space, then $(X, \delta(P))$ is normal.

A fuzzy pseudo-metric space holds less properties than a metric space in the topological sense. For example, we can define a pseudo-metric in $\widetilde{R}$ such that:

(1) If $x_a \ge y_b$, then $P(x_a, y_b) = 0$;

(2) $P(x_a, y_b) \le P(x_a, u_c) + P(u_c, y_b)$;

(3) If $A \ne \varnothing, B \ne \varnothing$, then $P(x_a, y_d) = |x - y|$;

(4) $\exists u_c$, $x_c \hat{q} u_c$, $P(y_b, u_c) < r \Leftrightarrow \exists v_d$, $y_b \hat{q} v_d$, $P(x_a, v_d) < r$.

Then an associated neighborhood of $x_a (a > 0)$ is $B = \{y_b : |x - y| < r\} (b > 0)$. It is a fuzzy open interval. It will generate a fuzzy topological space $(\widetilde{R}, d)$, discussed in Section 2.5.6. According to the previous analysis, it is not an induced space of *R* since every fuzzy open interval is also closed.

2.5.9 Induced fuzzy Euclidean space $\widetilde{R}^2$

We define a fuzzy topological space $(\widetilde{R}^2, \delta)$ such that it is induced from the Euclidean space R^2 with usual topology. If there is no confusion, we simplify it as $\widetilde{R}^2$. In $\widetilde{R}^2$, every fuzzy point $p(x_a, y_b)$ is closed, since every $p(x, y)$ in R^2 is closed. $\widetilde{R}^2$ is T_0, T_1, T_2, T_3, T_4 since it is induced and R^2 is normal. $\widetilde{R}^2$ is also connected.

Define a fuzzy Euclidean distance between two fuzzy points $p_a(x, y)$ and $q_b(u, v)$ in $\widetilde{R}^2$ by $d(p_a, q_b) = \sqrt{(x-u)^2 + (y-v)^2}$. Define a fuzzy open disk D_p^o of a fuzzy point $p_a(x, y)$ in $\widetilde{R}^2$ by $D_p^o = \{q_b : d(p_a, q_b) < r, 0 < b \le 1\}$. Define a fuzzy closed disk D_p of a fuzzy point $p_a(x, y)$ in $\widetilde{R}^2$ by $D_p = \{q_b : d(p_a, q_b) \le r, 0 < b \le 1\}$.

In general a fuzzy open disk may be not open in $\widetilde{R}^2$. Nor may a fuzzy closed disk be closed. For example, a fuzzy closed disk depicted in Figure 2.13 is not closed. In Figure

2.14 the membership values of a fuzzy closed disk are shown in the *I* axis.

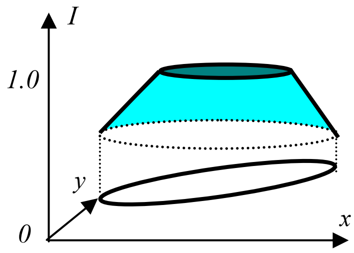


Figure 2.14 A fuzzy closed disk is maybe not closed in $\widetilde{R}^2$

According to the definition of the induced space and lower semicontinuity and upper semicontinuity, in $\widetilde{R}^2$ a fuzzy open disk is open iff it is a lower semicontinuous mapping from R^2 to [0,1]. A fuzzy closed disk is closed iff it is an upper semicontinuous mapping from R^2 to [0,1]. In the examples of Figure 2.15, *A* is an open set, and *B* is a closed set.

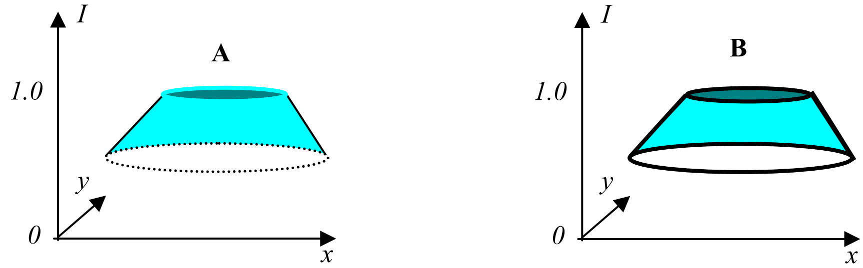


Figure 2.15 Open set and closed set in $\widetilde{R}^2$

In general, the interior and the closure of a fuzzy set *A* in $\widetilde{R}^2$ can be calculated by Proposition 2.13:

(1) $A^- = \bigvee_{0<\alpha\leq 1}(\alpha \wedge (A_\alpha)^-(x))$;

(2) $A^o = \bigvee_{0<\alpha\leq 1}(a \wedge (A_\alpha)^o(x))$.

Chapter Three

Topological Relations

in a Crisp Fuzzy Topological Space

3.1 Introduction

How to model spatial features is an essential question in GIS. An important issue is to understand the relationships between spatial features. Of all the relationships, topological relations play a fundamental role in GIS modeling. A query such as "who are my neighbors?" belongs to this province. Several approaches have been proposed for identifying topological relations between crisp spatial objects. Corbett (1979) introduced the algebraic topological structure for cartographic modeling. Allen (1983) identified 13 topological relations between two temporal intervals. The breakthrough on topological relations between spatial objects was made by the well-known 4-intersection and 9-intersection approaches proposed by Egenhofer and Franzosa (1991). A lot of research has been done based on this aspect (Egenhofer and Herring 1990a, 1990b, Herring 1991, Egenhofer *et al*. 1994a, 1994b, Egenhofer and Franzosa 1994, van Oosterom 1997, Molenaar 1996, Chen *et al*. 2001). On the other hand, Kainz *et al*. (1993) investigated the topological relations from the perspective of poset and lattice theory. Randell *et al*. (1992) described topological relations by using their RCC (Region Connection Calculus) theory, which is based on logic.

However, spatial objects are not always crisp. There are many fuzzy objects in reality, such as downtown area, forest and grassland. Fisher (1996) provides a good example of fuzzy objects by analyzing land cover classification on satellite images. Since Zadeh introduced fuzzy set theory in 1965, it has been widely researched theoretically, and successfully applied in many fields such as automatic control. A lot of research has been done using fuzzy mathematics for GIS models, for example Burrough (1989), Cheng *et al.* (1997), Brown (1998), and Cross and Firat (2000).

For GIS applications, understanding the topological relations between fuzzy spatial objects is vital when modeling fuzzy spatial objects. In this respect, several models have been proposed for modeling topological relations between fuzzy spatial objects (Clementini and Di Felice 1996, Cohn and Gotts 1996, Cohn *et al.* 1997, Dijkmeijer and Hoop 1996, Molenaar 1996, Zhan 1997, Winter 2000). Two of them, namely *the algebraic model* and *the egg-yolk model*, are described in detail for theoretic fuzzy modeling. The former model, proposed by Clementini and Di Felice (1996), is based on algebraic topology. In this model, a fuzzy region is defined as the union of two parts: the core region with a broad boundary (Figure 3.1). The definition of a fuzzy region is also discussed by Schneider (1999). The interior and the exterior of the region are assumed as open sets, while the broad boundary is a closed set. By using the 9-intersection approach, 44 different relations are identified in R^2.

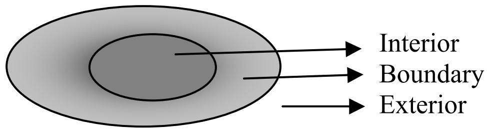


Figure 3.1 A region with a broad boundary (Clementini and Di Felice 1996)

The egg-yolk model was introduced by Cohn and Gotts (1996) and Cohn *et al.* (1997). In its simplest case, a region (egg) is composed of the inner subregion (yolk) and the outer subregion (white). The egg is the maximal extent of a vague region and the yolk is its minimal extent, while the white is the area of indeterminacy. Figure 3.2 shows the primitives of this model. Based on the RCC, 46 relations are identified by using so-called limits on the possible "complete crispings" or precise versions of a vague region. Any acceptable complete crisping must lie between the inner and outer limits of the defined yolk and egg.

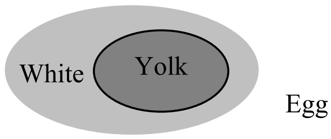


Figure 3.2 Primitives of the egg-yolk model (Cohn and Gotts 1996)

The above two models identify the topological relations between two fuzzy regions. Although the egg-yolk model is based on logic, it can also be perceived that the connection that the model adopted for the relations has the characteristics of topology (the connection holds when closures of two crisp regions share points). The algebraic model directly assumes the open set and the boundary set to be the two parts of a fuzzy region algebraically. Since an ordinary topological space does not allow the existence of any fuzzy set, it is impossible to imbed any fuzzy set in any ordinary topological space. In this sense the open set in the algebraic model is an algebraic assumption of a fuzzy region in the ordinary topological space.

An alternative way to identify the topological relations between fuzzy sets is to analyze them in fuzzy topological space. A question then arises of how to identify the topological relations between fuzzy sets in fuzzy topological space. The 9-intersection approach is well-known for the identification of the topological relations between crisp regions in the ordinary topological space. Is it possible to extend this approach for the identification between fuzzy spatial objects?

This chapter aims at identifying the topological relations between simple fuzzy regions in a special fts (Tang *et al.* 2003a). After discussions on the 9-intersection approach, a possible way to extend this approach is established in a crisp fuzzy topological space. The structure of this chapter is as follows. Section 3.2 selects the fuzzy boundary for GIS applications, based on several definitions of fuzzy boundary in fuzzy topological space. Section 3.3 analyzes the 9-intersection approach that will be updated in the thesis. Section 3.4 introduces a crisp fuzzy topological space and reveals some properties of this space. Section 3.5 introduces three intersection matrices in this space. Section 3.6 introduces the definition of a simple fuzzy region in this space. Section 3.7 identifies the topological relations between two simple fuzzy regions. Conclusions and discussions complete this chapter.

3.2 Fuzzy boundary

3.2.1 Definitions of fuzzy boundary

In GIS one of the most important notions we are concerned with is the fuzzy boundary. In fuzzy topology, several definitions have been proposed (Warren 1977, Pu and Liu 1980, Wu and Zheng 1991, Cuchillo-Ibáñez and Tarrés 1997). Let *A* be a fuzzy set in fts (X,δ):

(1) Warren (1977): The fuzzy boundary of a fuzzy set *A* is the infimum of all closed fuzzy sets *D* in *X* with the property $D(x) \geq A^{-}(x)$ for all $x \in X$ for which $(A^{-} \cap A^{c-})(x) > 0$.
(2) Pu and Liu (1980): The fuzzy boundary of a fuzzy set *A* is the intersection of the closure with the closure of the complement of a fuzzy set, *i.e.*, $\partial A = A^{-} \cap A^{c-}$. This form is identical to the boundary defined in the crisp space.
(3) Cuchillo-Ibáñez and Tarrés (1997): The fuzzy boundary of a fuzzy set *A* is the infimum of all closed fuzzy sets *D* in *X* with the property $D(x) \geq A^{-}(x)$ for all $x \in X$ for which $A^{-}(x) > A^{o}(x)$.

3.2.2 Selection of fuzzy boundary

In the cts, the boundary ∂A of a subset *A* is the intersection of the closure of *A* with the closure of the complement of *A*. It has the following properties:

Proposition 3.1 *In the cts, the boundary of a subset holds the following properties:*
(1) *It is closed;*

(2) *It is the difference between the closure and the interior of the set;*
(3) *It is the intersection of the closure with the closure of the complement of the set.*

It can be easily seen that the above definitions of fuzzy boundary are equivalent to the boundary of a crisp subset in the cts if all membership functions of the fuzzy sets are restricted to being characteristic functions. In the fts, however, these three properties cannot hold simultaneously for the fuzzy boundary, *i.e.*, $A^- \cap A^{c-} = A^- - A^o$ may not hold. This is due to $A \cap A^c \neq \emptyset$ in fuzzy set theory.

It is necessary to select one of the definitions for GIS applications. Let us call the boundaries defined by Warren, Pu and Liu, and Cuchillo-Ibáñez and Tarrés *boundary I* (∂A_I), *boundary II* (∂A_{II}) and *boundary III* (∂A_{III}), respectively. These definitions meet different properties of the three properties listed in Proposition 3.1. Boundary I satisfies Proposition 3.1(1), boundary II satisfies (1) and (3), and boundary III satisfies (1). Besides this, the following proposition is obvious and shows the relationships between these definitions.

Proposition 3.2 *Let A be a fuzzy set in fts* (X, δ). *Then*

(1) $A^- = \partial A_I \cup A^o = \partial A_{III} \cup A^o$
(2) $A^- \supseteq \partial A_{II} \cup A^o$
(3) $A^- \supseteq \partial A_I \supseteq \partial A_{III}$
(4) $\partial A_I \supseteq \partial A_{II}$

The difference between the three boundaries can be illustrated by an example. Let us construct an fts $(\widetilde{R}, \delta)$ that is induced from the Euclidean space *R*, *i.e.*, the elements of δ are exactly the family of all the lower semicontinuous mappings from *R* to $[0,1]$. A fuzzy open interval: $D^o = (x_a, y_b), 0 < a, b \leq 1, x < y$ is open if it is a lower semicontinuous mapping from *R* to $[0,1]$. It can be calculated by Proposition 2.13. A fuzzy closed interval ($D = [x_a, y_b], 0 < a, b \leq 1, x < y$) is closed if it is an upper semicontinuous mapping from *R* to $[0,1]$). This space is different from the space $(\widetilde{R}, \gamma)$ that is defined in Section 2.5.6, in that in $(\widetilde{R}, \delta)$ some fuzzy closed intervals defined in $(\widetilde{R}, \gamma)$ are not closed in $(\widetilde{R}, \delta)$, and in $(\widetilde{R}, \delta)$ some fuzzy open intervals defined in $(\widetilde{R}, \gamma)$ are not open in $(\widetilde{R}, \delta)$. In Figure 3.3 $D = [-2_a, 2_b]$ is closed in $(\widetilde{R}, \delta)$ and $(\widetilde{R}, \gamma)$. $D' = [-2_a, 2_b]$ is closed in $(\widetilde{R}, \gamma)$ but it is not closed in $(\widetilde{R}, \delta)$. The reason is explained in Section 2.5.6.

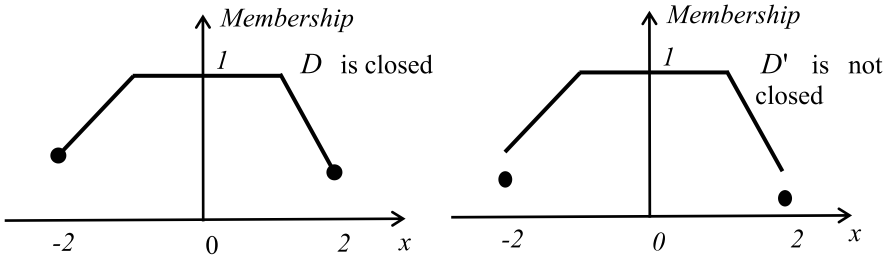


Figure 3.3 Fuzzy closed intervals in $(\widetilde{R},\delta)$

Suppose there is a fuzzy closed interval A in fts $(\widetilde{R},\delta)$ ($\varepsilon > 0$):

$$A = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = \varepsilon & if \quad |x| = 2 \\ A(x) = \big||x|-2\big| & if \quad 1 < |x| < 2 \\ A(x) = 1 & if \quad |x| = 1 \\ A(x) = 1 & if \quad |x| < 1 \end{cases}$$

In $(\widetilde{R},\delta)$, A is closed since it is an upper semicontinuous mapping from R to $[0,1]$. The boundaries are:

$$\partial A_I = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = \varepsilon & if \quad |x| = 2 \\ A(x) = \big||x|-2\big| & if \quad 1 < |x| < 2 \\ A(x) = 1 & if \quad |x| = 1 \\ A(x) = 0 & if \quad |x| < 1 \end{cases}, \quad \partial A_{II} = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = \varepsilon & if \quad |x| = 2 \\ A(x) = \big||x|-1\big| & if \quad 1 < |x| < 1.5 \\ A(x) = \big|2-|x|\big| & if \quad 1.5 \le |x| < 2 \\ A(x) = \varepsilon & if \quad |x| = 1 \\ A(x) = 0 & if \quad |x| < 1 \end{cases},$$

$$\partial A_{III} = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = \varepsilon & if \quad |x| = 2 \\ A(x) = 0 & if \quad 1 < |x| < 2 \\ A(x) = 1-\varepsilon & if \quad |x| = 1 \\ A(x) = 0 & if \quad |x| < 1 \end{cases}$$

Figure 3.4 illustrates these boundaries of A. Boundary I is shown by red dashed lines (with linked dots), boundary II is marked with green lines (with linked dots), and boundary III is marked with black dots.

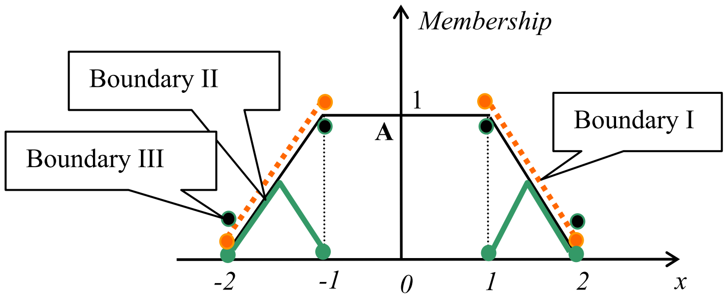


Figure 3.4 The different boundaries of fuzzy closed disk in $(\widetilde{R},\delta)$

In GIS applications, an intuitive understanding of fuzzy objects considers a point whose membership degree is less than 1 to be located near or at the rim of the object. The algebraic model (Clementini and Di Felice 1996) defines the broad boundary in the cts in this way. To meet such an intuitive condition, boundary I is the most suitable for GIS applications. According to Proposition 3.2(3) and (4), this boundary is greater than all other boundaries. This is reasonable because the other boundaries may not include a point whose membership degree is less than 1.

3.2.3 Properties of fuzzy boundary

The above example illustrates that Warren's definition is most suitable in GIS applications. We adopt his definition in this thesis, and denote it by ∂A. Several propositions related to the fuzzy boundary are listed for later discussions (Warren 1977).

Proposition 3.3 *Let A be a fuzzy set in fts* (X,δ).

(1) ∂A *is a closed set, and* $\partial A \subseteq A^{-}$;

(2) $\partial A = \varnothing$ *iff* A *is open, closed and crisp*;

(3) $A^{-} = A^{o} \cup \partial A = A \cup \partial A$;

(4) $\partial A \supseteq A^{-} \cap (X-A)^{-} \supseteq A^{-} - A^{o}$;

(5) $\partial(A^{o}) \cup \partial(A^{-}) \subseteq \partial A$;

(6) $\partial(\partial A) \subseteq \partial A$.

It is obvious that the boundary of the empty set and of the universe is empty according to Proposition 3.3(2).

3.3 Analysis of existing 9-intersection approach

3.3.1 9-intersection approach

The 9-intersection approach is derived from the cts for crisp sets. In classical set theory, between subset A and its complement A^{c} we have $A \cap A^{c} = \varnothing$ and $A \cup A^{c} = X$.

In the cts, similar properties hold: $A^o \cup \partial A = A^-$ and $A^o \cap \partial A = \varnothing$. Furthermore, the exterior, the boundary and the interior of a subset in the cts are mutually disjoint. These three concepts are called the *topological parts* of a subset. The 4-intersection and 9-intersection approaches utilize this property for formalizing topological relations between two crisp regions (Egenhofer and Herring 1990a). The former is defined by the intersections between the interior and the boundary of two sets. The latter is the extension of the 4-intersection in which the exterior is added into the intersection matrix.

9-intersection matrix in crisp topological space: *Let A, B be two subsets in cts X.*

$$I_9(A,B) = \begin{bmatrix} A^o \cap B^o & A^o \cap \partial B & A^o \cap B^e \\ \partial A \cap B^o & \partial A \cap \partial B & \partial A \cap B^e \\ A^e \cap B^o & A^e \cap \partial B & A^e \cap B^e \end{bmatrix} \quad (0)$$

By using the 4-intersection or 9-intersection approach, the relations between spatial objects can be identified based on topological properties in the intersections. Define a simple region as a regular closed set (that is $A^{o-} = A$) where A^o and ∂A are connected. Eight relations, *disjoint, contains, inside, equal, meet, covers, coveredby and overlap*, between two simple regions, are identified in R^2 by using the topological properties *empty* and *non-empty* dichotomy in the intersections (Egenhofer and Franzosa 1991).

3.3.2 More possible intersections

The 9-intersection approach adopts two important facts:

(1) The interior, boundary and exterior of a subset are topological invariants and they are mutually disjoint;

(2) The contents of the intersections between these three topological parts of two subsets are topological invariants.

The topological relations can then be identified by examining the topological invariants of the intersections. Although there are alternative topological invariants that will derive other topological relations (for example, the cardinal number and the dimension), the empty/non-empty invariant is the simplest set-oriented topological invariant in the cts.

In the cts, it is possible to generate more disjoint topological parts for a crisp subset. Considering the boundary of a subset in the cts, it can be decomposed into disjoint subsets such as the boundary of the boundary and the interior of the boundary. Even in the connected cts, the interior of the boundary could be a non-empty set. A sufficient condition that the interior of the boundary of a subset is empty is that the subset is regular closed (This is because if $A = A^{o-}$, then $(\partial A)^o = (A^- \cap A^{c-})^o$ $= A^{-o} \cap A^{c-o} = A^{-o} \cap A^{-c} \subseteq A^- \cap A^{-c} = \varnothing$.). If the subset is not regular closed, then the decomposition of the boundary of the subset is feasible. This property provides the possibility of decomposing more topological parts from a subset in the cts. For example, we can consider the four parts of a subset: the interior, the boundary of the boundary, the interior of the boundary and the exterior, which are mutually disjoint. If these four topological parts are non-empty, then we can derive a 4*4-intersection matrix between these parts of two subsets in the cts.

For GIS applications in which crisp subsets are considered, the 9-intersection approach is exhaustive in terms of the topological parts for identifying topological relations, owing to the assumption that a spatial region is a regular closed set. Under this assumption, the interior of the boundary is empty. It is also obvious that the other topological parts cannot be further decomposed. So it is not necessary to decompose three topological parts into more.

3.3.3 Limitations of 9-intersection approach in fts

The crisp topological space cannot accommodate any fuzzy set since all subsets should be crisp. One way of identifying topological relations between fuzzy sets is to accommodate them in an fts. Since in fuzzy set theory the properties $A \cap A^c = \varnothing$ and $A \cup A^c = X$ do not generally hold, correspondingly in an fts $A^o \cup \partial A = A^-$ and $A^o \cap \partial A = \varnothing$ cannot generally hold. The intersections between $A^o, \partial A, A^e$ could be non-empty, and the union of these three parts could be not equal to *X*. This leads to the fact that the identification of topological relations based on the 9-intersection approach will be not unique. For example, for two fuzzy sets A, B in an fts *X* there is a 9-intersection:

$$\begin{bmatrix} A^o \cap B^o = \varnothing & A^o \cap \partial B \neq \varnothing & A^o \cap B^e \neq \varnothing \\ \partial A \cap B^o = \varnothing & \partial A \cap \partial B \neq \varnothing & \partial A \cap B^e \neq \varnothing \\ A^e \cap B^o \neq \varnothing & A^e \cap \partial B \neq \varnothing & A^e \cap B^e \neq \varnothing \end{bmatrix}$$

This intersection could be the result where $A^o \cap \partial A \neq \varnothing$ and $B^o \cap \partial B = \varnothing$ or $A^o \cap \partial A = \varnothing$ and $B^o \cap \partial B = \varnothing$.

3.4 A crisp fuzzy topological space

3.4.1 Introduction and definition

The problem of the 9-intersection approach in the fts arises because of the non-disjointness property of the three topological parts in the fts. To see under what condition they will be disjoint, we have the following proposition:

Proposition 3.4 *Let A be a fuzzy set in fts* (X, δ). *Then* $\partial A \cap A^o = \varnothing$ *iff* A^o *is crisp*.

Proof:

$\Rightarrow$ If $A^o(x) = 0$, then $(X - A^o)^-(x) = 1$.

If $A^o(x) > 0$, then $\partial A(x) = 0$ since $(\partial A \cap A^o)(x) = 0$. Since $A^{o-}(x) > 0$ and $(A^{o-} \cap (X - A^o)^-)(x) = 0$, thus $(X - A^o)^-(x) = 0$. So $A^o(x) = 1$.

$\Leftarrow$ Assume that A^o is crisp. If $(A^{o-} \cap (X - A^o)^-)(x) > 0$, then $(X - A^o)^-(x) = 1 \geq A^-(x)$, so $\partial A \subseteq (X - A^o)$. Hence when $A^o(y) = 1$, $\partial A(y) = 0$.

The above proposition reveals that, if and only if the open set of a subset in the fts is crisp, will the interior and the boundary of the set then be disjoint. We can then analyze the topological relations in a special fuzzy topological space.

Definition 3.5 *The fuzzy topological space* (X, C) *is called a crisp fuzzy topology space (or a crisp fts in short) if all open sets in* (X, C) *are crisp.*

Under such fts, all closed sets are also crisp; therefore all closures are also crisp. The difference between a crisp fts and a cts lies in the fact that the crisp fts allows the existence of fuzzy sets while the cts does not. However, in (X, C), since all open sets are crisp, the topology *C* on *X* is crisp. The basic properties of (X, C) are the same as cts, as we will show in the following.

Proposition 3.6 *The fuzzy boundary of a fuzzy set A in* (X, C) *is the intersection of the closure of the fuzzy set with the closure of the complement of the fuzzy set:* $\partial A = A^- \cap A^{c-}$.

Proof: The definition of the fuzzy boundary can be written as the closure of $\{x : (A^- \cap A^{c-})(x) > 0\}$. In (X, C), $(A^- \cap A^{c-})(x) > 0$ becomes $(A^- \cap A^{c-})(x) = 1$ and is still a closed set. So $A^- \cap A^{c-} \supseteq \partial A$. On the other hand, from Proposition 3.3(4), $\partial A \supseteq A^- \cap (X - A)^-$, so $\partial A = A^- \cap A^{c-}$.

Corollary 3.7 *In (X,C), the boundary and the interior of any fuzzy set are disjoint*: $\partial A \cap A^o = \varnothing$.

Proof:

$\partial A \cap A^o = A^- \cap A^{c-} \cap A^o$ (Proposition 3.6)

$= A^o \cap A^{c-}$ (According to Proposition 2.7(1) and 2.8(1), $A^o \subseteq A^-$)

$\subseteq A^o \cap A^{oc-}$ (According to Proposition 2.7(1), $A^o \subseteq A$, then $A^{oc} \supseteq A$)

$= A^o \cap A^{oc}$ (*A* is open)

$= \varnothing$ (*A* is crisp)

In the later proofs, we will simplify the process without showing the propositions.

It also can be easily proven by the above procedure that in (X, C) the boundary, the interior and the exterior of a fuzzy set are mutually disjoint.

In the cts, there is a theorem stating that, if both *A* and *B* are open or both are closed, then $A - B$ is separated from $B - A$. This theorem cannot hold in the general fts.

However, in (X,C), it still holds. Specifically, we have:

Proposition 3.8 *In* (X,C)*, if* $A^o, \partial A$ *and* A^e *of fuzzy set A are not empty, then* A^o *and* A^e *are Q-separated.*
Proof: Since $A^o \cup \partial A = A^-$, and $A^o, \partial A$, A^e are all crisp, then $A^- \cap A^e = \varnothing$. Similarly, $A^e \cup \partial A = A^{e-}$, then $A^{e-} \cap A^o = A^{-oc} \cap A^o = \varnothing$.

Since A^o and A^e are Q-separated and crisp, then A^o and A^e are separated according to Proposition 2.10. Then in (X,C) Q-neighborhood and neighborhood imply each other. Therefore we have:

Proposition 3.9 *If* (X,C) *is connected, then there are no two disjoint open or two disjoint closed subsets A, B where* $A \cup B = X$.

Proposition 3.9 reveals that in (X,C) the open-connectedness and the closed-connectedness of a fuzzy set imply each other. Therefore we can just use the connectedness of a fuzzy set to replace the open-connectedness and the closed-connectedness.

3.4.2 Fuzzy boundary in crisp topological space

In Section 3.3.2 it is mentioned that in cts the boundary can be decomposed into more topological parts. In (X,C), the boundary of a subset may also have its interior and its boundary of the boundary. On the other hand, the interior and the closure of a subset also have their boundaries. The following proposition reveals the relationship between the boundary of the boundary and the boundary of the interior with the boundary of the closure in (X,C).

Proposition 3.10 *The boundary of the boundary of a fuzzy set A in* (X,C) *is the union of the boundary of the closure and the boundary of the interior of a fuzzy set, i.e.,* $\partial(\partial A) = \partial(A^-) \cup \partial(A^o)$.
Proof:

$$\begin{aligned}\partial(\partial A) &= (\partial A)^- \cap (\partial A)^{c-} = \partial A \cap (\partial A)^{c-} = \partial A \cap (A^- \cap A^{c-})^{c-} = \partial A \cap (A^{-c} \cup A^{c-c})^- \\ &= \partial A \cap (A^{-c-} \cup A^{c-c-}) = \partial A \cap (A^{-c-} \cup A^{o-}) = (\partial A \cap A^{-c-}) \cup (\partial A \cap A^{o-}) \\ &= (A^- \cap A^{c-} \cap A^{-c-}) \cup (A^- \cap A^{oc} \cap A^{o-}) = (A^- \cap A^{-c-}) \cup (A^{o-} \cap A^{oc}) \\ &= (A^{--} \cap A^{-c-}) \cup (A^{o-} \cap A^{oc-}) = \partial(A^-) \cup \partial(A^o)\end{aligned}$$

Proposition 3.11

(1) *In* (X,C)*, the boundary is the union of the interior of the boundary, the boundary of the interior and the boundary of the closure of a fuzzy set A:* $\partial A = (\partial A)^o \cup \partial(A^-) \cup \partial(A^o)$;

(2) *In* (X,C)*, the interior of the boundary and the boundary of the boundary of*

a fuzzy set A are disjoint: $(\partial A)^o \cap \partial(\partial A) = \varnothing$.

Proof:

(1) $\partial A = (\partial A)^- = (\partial A)^o \cup \partial(\partial A) = (\partial A)^o \cup \partial(A^-) \cup \partial(A^o)$

(2) $(\partial A)^o \cap \partial(A^-) = (A^- \cap A^{c-})^o \cap (A^{--} \cap A^{-c-}) = A^{-o} \cap A^{c-o} \cap A^- \cap A^{-c-}$

$$= A^{-o} \cap A^{c-o} \cap A^{-c-} \subseteq A^{-o} \cap A^{-c-} = A^{-o} \cap A^{-oc} = \varnothing$$

In Propositions 3.10 and 3.11, the boundary of the boundary of a fuzzy set is introduced, and the boundary can be broken into the interior of the boundary and the boundary of the boundary. A natural question arises: "Can we decompose the boundary into more topological parts, such as the boundary of the boundary of the boundary and the interior of the boundary of the boundary?" We will prove that the boundary of the boundary of a fuzzy set *A* is equal to the boundary of the boundary of the boundary in (X, C) .

Proposition 3.12 *In* (X, C) *, the boundary of the boundary of the boundary is equal to the boundary of the boundary of a fuzzy set A:* $\partial(\partial(\partial A)) = \partial(\partial A)$.

Proof: $\partial(\partial(\partial A)) = \partial(\partial A) \cap (X - \partial(\partial A))^-$.

Since $(X - \partial(\partial A))^- = (A^o \cup A^{-c} \cup (\partial A)^o)^- = A^{o-} \cup A^{-c-} \cup (A^{-o} \cap A^{c-o})^-$;

and $X - A^{o-} \cup A^{-c-} = (X - A^{o-}) \cup (X - A^{-c-}) = A^{o-c} \cap A^{-c-c} = A^{c-o} \cap A^{-o}$;

so

$$A^{o-} \cup A^{-c-} \cup (A^{-o} \cap A^{c-o})^-$$
$$\supseteq (A^{o-} \cup A^{-c-}) \cup (A^{-o} \cap A^{c-o}) = (A^{o-} \cup A^{-c-}) \cup (X - (A^{o-} \cup A^{-c-})) = X \, .$$

Therefore $\partial(\partial(\partial A)) = \partial(\partial A) \cap (X - \partial(\partial A))^- = \partial(\partial A)$

Corollary 3.13 *In* (X, C) *, the interior of the boundary of the boundary of a fuzzy set is empty:* $(\partial(\partial A))^o = \varnothing$.

Corollary 3.13 shows that in (X, C) the decomposition of the boundary of a fuzzy set into the boundary of the boundary and the interior of the boundary that are mutually disjoint is exhaustive, since the interior of the boundary of the boundary of a fuzzy set is empty. Based on Proposition 3.11(1), it is also obvious that the exterior and the interior cannot be decomposed any more in (X, C) .

3.4.3 Summary of boundaries in different topological spaces

We summarize some conclusions about the boundaries in different spaces. In Table 3.1 (X, τ) , (X, δ) and (X, C) represent a cts, a general fts and a crisp fts, respectively. Table 3.1 shows that the properties of the fuzzy boundary in (X, C) are the same as those in (X, τ) .

Table 3.1 Summary of boundary's properties in different topological spaces

Spaces / Boundary	Crisp topological space (X,τ)	General fuzzy topological space (X,δ)	Crisp fuzzy topological space (X,C)
Boundary of a subset *A*	$\partial A = A^- \cap A^{c-}$	$\partial A \supseteq A^- \cap A^{c-}$	$\partial A = A^- \cap A^{c-}$
Boundary of the boundary	$\partial(\partial A) = \partial(A^o) \cup \partial(A^-)$		$\partial(\partial A) = \partial(A^o) \cup \partial(A^-)$
Decomposition of the boundary	$\partial A = (\partial A)^o \cup \partial(\partial A)$ $\partial A = \partial(A^o) \cup (\partial A)^o \cup \partial(A^-)$	$\partial A = (\partial A)^o \cup \partial(\partial A)$ $\partial A \supseteq \partial(A^o) \cup (\partial A)^o \cup \partial(A^-)$ [1]	$\partial A = (\partial A)^o \cup \partial(\partial A)$ $\partial A = \partial(A^-) \cup (\partial A)^o \cup \partial(A^o)$
Comparison between the boundary and the boundary of the boundary	$\partial(\partial A) \subseteq \partial A$	$\partial(\partial A) \subseteq \partial A$	$\partial(\partial A) \subseteq \partial A$
Comparisons between the boundary of the boundary and the boundary of the boundary of the boundary	$\partial(\partial(\partial A)) = \partial(\partial A)$	$\partial(\partial(\partial A)) \subseteq \partial(\partial A)$ [2]	$\partial(\partial(\partial A)) = \partial(\partial A)$
Relationships between the interior, the boundary and the exterior	$A^o \cap \partial A = \varnothing$ $A^e \cap \partial A = \varnothing$ $A^o \cap A^e = \varnothing$ $(\partial A)^o \cap \partial(\partial A) = \varnothing$	The contents of these intersections are uncertain.	$A^o \cap \partial A = \varnothing$ $A^e \cap \partial A = \varnothing$ $A^o \cap A^e = \varnothing$ $(\partial A)^o \cap \partial(\partial A) = \varnothing$

3.5 Intersection matrices in crisp fts

3.5.1 3*3-intersection matrix in crisp fts

Based on the above discussion, it is clear that the interior, the boundary and the exterior of a fuzzy set are mutually disjoint in a crisp fts (X,C). Therefore, we can formulate a 3*3-intersection matrix between these topological parts of two fuzzy sets *A* and *B* in (X,C).

[1] This is because $\partial A \supseteq \partial(A^o)$ and $\partial A \supseteq \partial(A^o) \cup \partial(A^-)$. So $\partial A \supseteq \partial(A^o) \cup (\partial A)^o \cup \partial(A^-)$.

[2] This is because $\partial(\partial A) \subseteq \partial A$.

3*3-intersection matrix in crisp fts: *Let A, B be two fuzzy sets in a crisp fts* (X, C).

$$I_{3*3}(A,B) = \begin{bmatrix} A^o \cap B^o & A^o \cap \partial B & A^o \cap B^e \\ \partial A \cap B^o & \partial A \cap \partial B & \partial A \cap B^e \\ A^e \cap B^o & A^e \cap \partial B & A^e \cap B^e \end{bmatrix} \tag{1}$$

We call (1) the *3*3-intersection matrix in* (X, C). According to the definition of topological relation, the fuzzy topological relation from fuzzy set *A* to fuzzy set *B* is a fuzzy relation that is topological invariant under a fuzzy homeomorphism. We now show that this intersection matrix can identify the topological relations from fuzzy set *A* to fuzzy set *B*. We also write the topological relation between *A* and *B*. Let $f^{\rightarrow}$ be a homeomorphic mapping from fts *X* to fts *Y*: $f^{\rightarrow}: X \rightarrow Y$. $f^{\rightarrow}: X \rightarrow Y$ is intersection preserving, that is, for two subsets *A* and *B* in *X*, $f^{\rightarrow}(A \cap B) = f^{\rightarrow}(A) \cap f^{\rightarrow}(B)$. Then $f^{\rightarrow}(A^o \cap B^o) = f^{\rightarrow}(A^o) \cap f^{\rightarrow}(B^o)$, $f^{\rightarrow}(A^o \cap \partial B) = f^{\rightarrow}(A^o) \cap f^{\rightarrow}(\partial B)$, and the same for the remaining intersections $\partial A \cap \partial B$, $\partial A \cap B^e$, $A^e \cap B^o$, $A^e \cap \partial B$, $A^e \cap B^e$. Therefore the topological relation from *A* to *B* can be identified by the topological invariants in the intersections. Note the two conditions listed in Section 3.3.2 are also necessary in the fts. If the fuzzy sets in the intersections are not topological properties, then the relation identified by the topological invariant in the intersections is not a topological relation from *A* to *B*. For example, let *A'*, *A"* be arbitrary subsets of *A* and *B', B"* be arbitrary subsets of *B*. Then the relation identified by the intersection $A' \cap B'$, $A'' \cap B''$ is not topological even if we adopt the topological properties in the intersections. Similarly, when the intersection content is not a topological invariant, even if the subsets in the intersection are topological properties, the relation is also not topological. For example, assume that the area size is the content of the intersection, the result relation is not topological.

Comparing intersection matrix (1) with (0), it can be found that they are of the same form. Actually intersection matrix (1) is the exact extension of the 9-intersection matrix (0) from a cts to (X, C). If all subsets are crisp, then (X, C) degenerates into a cts. Then 3*3-intersection matrix (1) also changes into matrix (0). This is the exact 9-intersection matrix that Egenhofer and Franzosa deduced in the cts in 1991.

The difference between this matrix and the algebraic model lies in the fact that the algebraic model is based on the cts, which "allows" the existence of the fuzzy set – "the broad boundary". In the intersection matrix (1), however, the fuzzy boundary is crisp. If the membership value of every element of the broad boundary is projected to 1 of interval [0,1], then the broad boundary is equal to the fuzzy boundary.

3.5.2 4*4-intersection matrix in crisp fts

The above 3*3-intersection matrix is derived based on the interior, boundary, and exterior. In Section 3.4, it has been shown that a fuzzy set can be decomposed into four parts: the interior, the boundary of the boundary, the interior of the boundary and the exterior, which are mutually disjoint; therefore, we can introduce a 4*4-intersection matrix in (X,C) between two fuzzy sets.

4*4-intersection matrix in crisp fts: *Let A, B be two fuzzy sets in a crisp fts* (X,C).

$$I_{4*4}=\begin{bmatrix} A^o\cap B^o & A^o\cap\partial(\partial B) & A^o\cap(\partial B)^o & A^o\cap B^e \\ \partial(\partial A)\cap B^o & \partial(\partial A)\cap\partial(\partial B) & \partial(\partial A)\cap(\partial B)^o & \partial(\partial A)\cap B^e \\ (\partial A)^o\cap B^o & (\partial A)^o\cap\partial(\partial B) & (\partial A)^o\cap(\partial B)^o & (\partial A)^o\cap B^e \\ A^e\cap B^o & A^e\cap\partial(\partial B) & A^e\cap(\partial B)^o & A^e\cap B^e \end{bmatrix} \quad (2)$$

We call (2) the *4*4-intersection matrix in* (X,C). According to the analysis in Section 3.5,1, this intersection matrix can also be adopted to identify the topological relations between two fuzzy sets in (X,C). Based on Proposition 3.11, it is clear that the 4*4-intersection matrix between two fuzzy sets is exhaustive in terms of boundaries, interiors, and exteriors in (X,C) if the disjointness between the boundary of the interior and the boundary of the closure is not considered. That is, besides these four topological parts, there are no more topological parts that are mutually disjoint. Therefore, if the disjointness between the boundary of the interior and the boundary of the closure is not considered, then the 4*4-intersection matrix is the formalization that will identify *all* topological relations between two fuzzy sets when a topological invariant is fixed.

In particular, if the boundary of the interior of a fuzzy set is equal to the boundary of the closure of that set in (X,C), then the interior of the boundary is empty. For a crisp simple region, the interior of the boundary is empty. Therefore, the relation between a fuzzy set and a crisp set in (X,C) can be formalized by a 4*3-intersection matrix:

$$I_{4*3}=\begin{bmatrix} A^o\cap B^o & A^o\cap\partial B & A^o\cap B^e \\ \partial(\partial A)\cap B^o & \partial(\partial A)\cap\partial B & \partial(\partial A)\cap B^e \\ (\partial A)^o\cap B^o & (\partial A)^o\cap\partial B & (\partial A)^o\cap B^e \\ A^e\cap B^o & A^e\cap\partial B & A^e\cap B^e \end{bmatrix} \quad (2')$$

3.5.3 5*5-intersection matrix in crisp fts

If the boundary of the closure and the boundary of the interior are disjoint, then the exterior, the boundary of the closure, the interior of the boundary, the boundary of the interior, and the interior of a fuzzy set are five mutually disjoint parts in (X,C). We can then form a 5*5-intersection matrix if two fuzzy sets hold this property.

5*5-intersection matrix in crisp fts: *Let A, B be two fuzzy sets in a crisp fts* (X,C).

$$I_{5*5} = \begin{bmatrix} A^o \cap B^o & A^o \cap \partial(B^o) & A^o \cap (\partial B)^o & A^o \cap \partial(B^-) & A^o \cap B^e \\ \partial(A^o) \cap B^o & \partial(A^o) \cap \partial(B^o) & \partial(A^o) \cap (\partial B)^o & \partial(A^o) \cap \partial(B^-) & \partial(A^o) \cap B^e \\ (\partial A)^o \cap B^o & (\partial A)^o \cap \partial(B^o) & (\partial A)^o \cap (\partial B)^o & (\partial A)^o \cap \partial(B^-) & (\partial A)^o \cap B^e \\ \partial(A^-) \cap B^o & \partial(A^-) \cap \partial(B^o) & \partial(A^-) \cap (\partial B)^o & \partial(A^-) \cap \partial(B^-) & \partial(A^-) \cap B^e \\ A^e \cap B^o & A^e \cap \partial(B^o) & A^e \cap (\partial B)^o & A^e \cap \partial(B^-) & A^e \cap B^e \end{bmatrix} \quad (3)$$

If the boundary of the interior and the boundary of the closure are disjoint for two fuzzy sets, then the 5*5-intersection matrix is exhaustive in terms of the interior, the boundary and the exterior. That is, the 5*5-intersection matrix will identify *all* topological relations between two fuzzy sets when a topological invariant is fixed. No more topological relations can be identified.

3.6 A definition of a simple fuzzy region in crisp fts

The above intersection matrices can identify topological relations between two fuzzy sets. However, not all fuzzy sets are fuzzy spatial objects. In order to define fuzzy spatial objects, we have to limit the general fuzzy sets by some conditions. In this section we introduce a definition of a simple fuzzy region in a crisp fts. Firstly, let us look at the possible forms of fuzzy regions in reality.

3.6.1 Simple fuzzy regions in GIS

When natural phenomena such as mountains, oceans, grassland and population distribution density are represented in GIS, there are two ways to represent them: crisp form and fuzzy form. Take the coastline as an example. In the crisp situation, the coastline is usually represented by an average tide. The attribute of an area object in terms of water is assigned 0 if the object is higher than the average tide, and the attribute is 1 if the object is lower than the average tide (Figure 3.5).

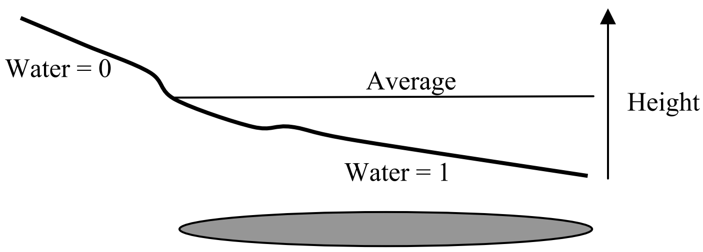


Figure 3.5 Representation of a crisp simple region in GIS

When we represent fuzziness related to the coastline, there are several options, depending on the requirement of concrete applications.

Fuzzy region *I*: modeling the fuzziness by a single value. For example, when we form a water area, the fuzziness of the object can be assigned in a simple way according to the low tide and the high tide. For example, the membership value of a fuzzy water area can be assigned as follows: the membership is 1 if the area is below low tide, 0.5 if it is between low tide and high tide, and 0 if it is higher than high tide (Figure 3.6).

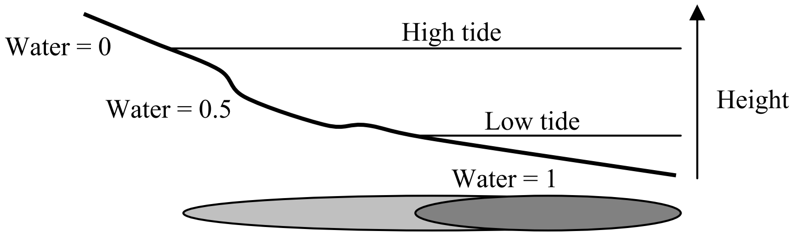


Figure 3.6 Representation of a water area by a simple membership

Fuzzy region *II*: modeling fuzziness in terms of finite membership values. In the coastline example, the membership values of a water area can also be represented by several membership values (Figure 3.7).

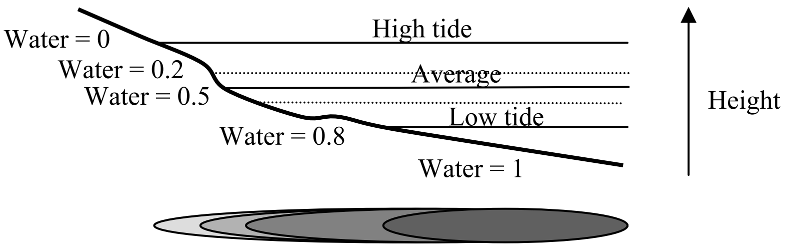


Figure 3.7 Representation of a water area by finite memberships

Fuzzy region *III*: the fuzziness of water can also be represented by the value of a continuous mapping from the height to [0,1] (Figure 3.8).

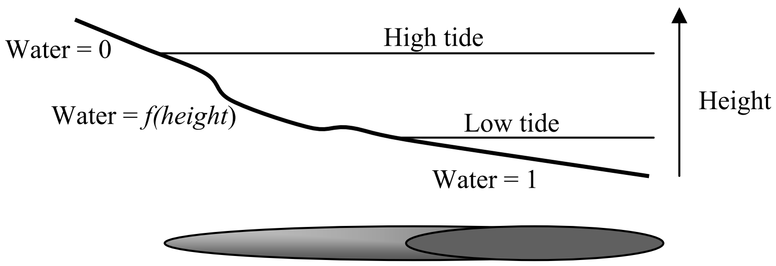


Figure 3.8 Representation of a water area by a membership function

3.6.2 A definition of a simple fuzzy region in crisp fts

A crisp simple region has been abstracted as a regular closed set with a connected interior in the connected crisp topological space. The definition of a fuzzy region has also been discussed by Schneider (1999). However, in his definition a fuzzy region is an open set, which is inconsistent with the definition of a crisp simple region. We now generalize a simple fuzzy region in (X,C). As we know, a crisp subset is a special case of fuzzy set. Therefore it is natural to consider that a crisp simple region is also a special case of a simple fuzzy region. We define a simple fuzzy region in (X,C):

Definition 3.14 *A fuzzy set A is called a simple fuzzy region in the connected* (X,C), *such that:*

(1) *The closure of A is a proper non-empty connected regular closed subset, i.e.,* $A^{-}=A^{-o-}$;
(2) *The support of A is equal to the closure;*
(3) *The interior A is a non-empty connected regular open set;*
(4) *The boundary of A, the interior of the boundary of A and the exterior of A are connected.*

The first condition is extended according to the requirement of a crisp simple region. In the case that *A* is a crisp simple region, it should be closed. The regularity will remove some "dangle points" or "break lines" in the closure of fuzzy set *A*. The second condition requires that *A* should be equal to a crisp simple region when *A* is projected to 1 of [0,1]. It means that we do not allow an open set (for instance an open disk) in (X,C) to be a simple fuzzy region. The third condition will remove some "dangle points" or "break lines" in the interior of fuzzy set *A*, and will allow only "one piece" for the interior. The fourth condition requires the boundary and the interior of the boundary of a simple fuzzy region to be also connected. If the interior of the boundary of *A* is empty and *A* is crisp, then *A* is a simple crisp region. The exterior should be connected otherwise a simple fuzzy region may contain a "hole".

Proposition 3.15 *In connected* (X,C), *the boundary of the interior and the boundary of the closure of a simple fuzzy region are not empty:* $\partial(A^{o})\neq\varnothing$; $\partial(A^{-})\neq\varnothing$.
Proof: Because (X,C) is connected, *X* cannot be the union of two disjoint open sets.

The following proposition is obvious according to Proposition 3.8.
Proposition 3.16 *Let A be a simple fuzzy region in* (X,C).

(1) *The pairs* A^{o} *and* $X-A^{o-}$, A^{o} *and* $(\partial A)^{o}$, A^{o} *and* A^{e} *are (Q)-separated;*
(2) *The pairs* $(\partial A)^{o}$ *and* $X-(\partial A)^{o-}$, $(\partial A)^{o}$ *and* A^{e} *are (Q-)separated.*

Proposition 3.17 *Let A and B be simple fuzzy regions in* (X,C).

(1) *If* $A^{o}\cap B^{o}\neq\varnothing, A^{o}\cap\partial(B^{o})=\varnothing$, *then* $A^{o}\subseteq B^{o}$;
(2) *If* $A^{o}\cap(\partial B)^{o}\neq\varnothing, A^{o}\cap\partial((\partial B)^{o})=\varnothing$, *then* $A^{o}\subseteq(\partial B)^{o}$.

Proof:

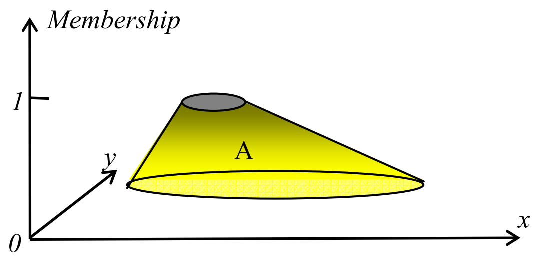
(1) $B^{o}\cup(X-B^{o-})=X-\partial(B^{o})$ because they are all crisp. Since $A^{o}\cap\partial(B^{o})=\varnothing$, $A^{o}\subseteq B^{o}\cup(X-B^{o-})$. Since B^{o} and $X-B^{o-}$ are

Q-separated, A^o is connected. Then either $A^o \subseteq B^o$ or $A^o \subseteq (X - B^{o-})$. Since $A^o \cap B^o \neq \varnothing$, $A^o \subseteq B^o$.

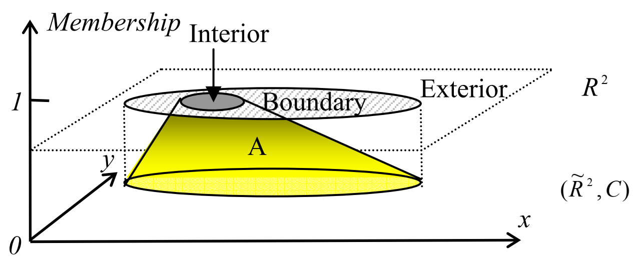
(2) $(\partial B)^o \cup (X - (\partial B)^{o-}) = X - \partial((\partial B)^o)$. Since $A^o \cap \partial((\partial B)^o) = \varnothing$, $A^o \subseteq (\partial B)^o \cup (X - (\partial B)^{o-})$. Then either $A^o \subseteq (\partial B)^o$ or $A^o \subseteq (X - (\partial B)^{o-})$ since $(\partial B)^o$ and $X - (\partial B)^{o-}$ are separated and A^o are connected. Since $A^o \cap (\partial B)^o \neq \varnothing$, $A^o \subseteq (\partial B)^o$.

In order to illustrate a simple fuzzy region, let X be a fuzzy Euclidean space $\widetilde{R}^2$. We define a crisp fts $(\widetilde{R}^2, C)$. The fuzzy Euclidean space will have the ordinary Euclidean distance if all points are crisp. C is the crisp fuzzy topology on $\widetilde{R}^2$ if every fuzzy open disk is open and crisp. The fuzzy topology C on $\widetilde{R}^2$ is equal to the usual topology on the Euclidean space R^2. Therefore, $(\widetilde{R}^2, C)$ is normal, p-normal and connected. $(\widetilde{R}^2, C)$ is not a stratified space. The difference between $(\widetilde{R}^2, C)$ and the ordinary Euclidean space R^2 lies in the fact that $(\widetilde{R}^2, C)$ allows fuzzy sets, but R^2 has no fuzzy sets.

The interpretation of a simple fuzzy region in $(\widetilde{R}^2, C)$ is illustrated in Figure 3.9. Figure 3.9(I) shows a simple fuzzy region in reality. Figure 3.9(II) shows the definition of a simple fuzzy region in $(\widetilde{R}^2, C)$ and some related concepts. In Figure 3.9(II), the membership values of a fuzzy region are represented in axis *membership*.



I: A simple fuzzy region in reality



II: A definition of a simple fuzzy region

Figure 3.9 A definition of a simple fuzzy region in $(\widetilde{R}^2, C)$

Figure 3.9 shows only a few properties of a simple fuzzy region. Figure 3.10 illustrates the definition in a planar way. Since in $(\widetilde{R}^2, C)$ all topological properties are crisp, it is possible to draw all concepts in the definition of a simple fuzzy region in the plane. In Figure 3.10 the closure is the definition of a simple fuzzy region from the reality.

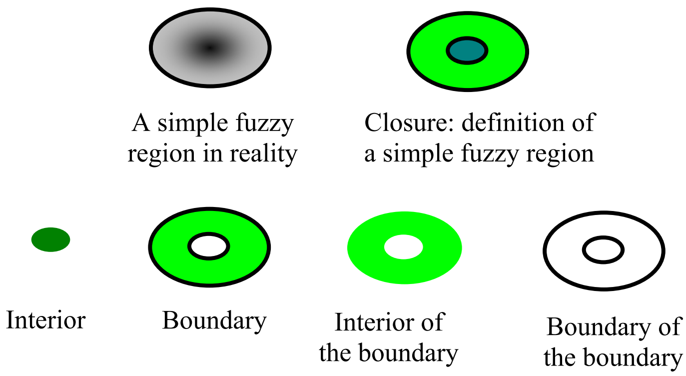


Figure 3.10 Interior, boundary, interior of the boundary and boundary of the boundary of a simple fuzzy region in $(\widetilde{R}^2, C)$

Figure 3.11 illustrates the possible settings of a simple fuzzy region in a planar way. Figure 3.12 gives examples of the impossible settings of a simple fuzzy region that do not meet all the four conditions.

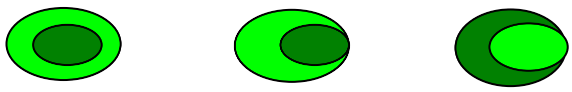


Figure 3.11 Possible settings of a simple fuzzy region in $(\widetilde{R}^2, C)$

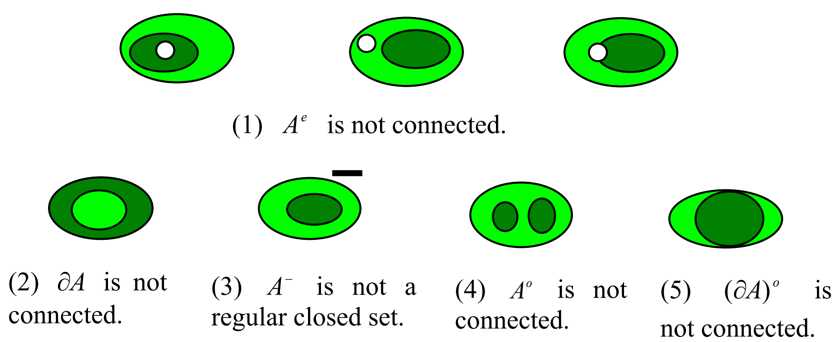


Figure 3.12 Impossible settings of simple fuzzy regions in $(\widetilde{R}^2, C)$

3.7 Topological relations between simple fuzzy regions in $(\widetilde{R}^2, C)$

3.7.1 Identification by 3*3-intersection matrix

All 3*3-intersection, 4*4-intersection and 5*5-intersection matrices can be applied as *approaches* for the identification of topological relations between two simple fuzzy regions in different settings. We focus on the identification of topological relations between two simple fuzzy regions in $(\widetilde{R}^2, C)$. A simple fuzzy region is further regarded as bounded and two-dimensional. The 3*3-intersection matrix can be adopted if the boundary of a fuzzy region is considered as one topological part. The 4*4-intersection matrix can be used when four topological parts are distinguished. Egenhofer and Herring (1990b) list 12 geometric conditions for relations between two simple regions in R^2. Clementini and Di Felice updated these 12 geometric conditions in their algebraic model for their fuzzy regions in R^2. Their conditions can be directly applied if the 3*3-intersection matrix is used for identifying topological relations between two simple fuzzy regions in $(\widetilde{R}^2, C)$, since the fuzzy topology *C* is equal to the topology on the Euclidean space R^2.

Let *A* and *B* be two simple fuzzy regions in $(\widetilde{R}^2, C)$. These conditions are:

(1) The exteriors of two fuzzy regions intersect each other;
(2) The boundary of *A* intersects at least one part of *B*, and vice versa;
(3) If the interior of *A* intersects the interior and the exterior of *B*, then it must also intersect the boundary of *B*, and vice versa;
(4) If both boundaries do not intersect each other, then at least one boundary must intersect its opposite exterior;
(5) If both boundaries intersect the opposite interiors, then the boundaries must also intersect each other;
(6) If the interior of *A* intersects the exterior of *B*, then the boundary of *A* must also intersect the exterior of *B*, and vice versa;
(7) If both interiors are disjoint and the boundary of *A* intersects the interior of *B*, then the two boundaries must intersect each other, and vice versa;
(8) If the interior of *A* is a subset of the closure of *B*, then the boundary of *A* must intersect the closure of *B*, and vice versa;
(9) If both interiors are disjoint, then the interior of *A* intersects either the boundary or the exterior of *B*, and vice versa;
(10) If the interior of *A* does not intersect the closure of *B*, then the boundary of *A* must intersect the exterior of *B*, and vice versa;
(11) If the boundary of *A* intersects the interior and the exterior of *B*, then it must also intersect the boundary of *B*, and vice versa;
(12) If the closure of *A* is a subset of the interior of *B*, then the exterior of *A* must intersect the interior of *B*, and vice versa.

Based on these conditions, 44 relations can be identified by using the 3*3-intersection matrix (Appendix 1). In Appendix 1, if the intersection between two fuzzy sets is empty, then it is 0, otherwise 1. For example, the disjointness relation between two fuzzy regions that is described by:

$$\begin{bmatrix} A^o \cap B^o = \varnothing & A^o \cap \partial B = \varnothing & A^o \cap B^e \neq \varnothing \\ \partial A \cap B^o = \varnothing & \partial A \cap \partial B = \varnothing & \partial A \cap B^e \neq \varnothing \\ A^e \cap B^o \neq \varnothing & A^e \cap \partial B \neq \varnothing & A^e \cap B^e \neq \varnothing \end{bmatrix}$$

is simplified as:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

where 0 represents the empty set, 1 otherwise. The result is the same as that of the algebraic model.

3.7.2 Identification by 4*4-intersection matrix

In general there are 2^{16}= 65,536 relations between two fuzzy sets by using the 4*4-intersection matrix. Twelve constraints are generalized for the identification of topological relations between two simple fuzzy regions in $(\widetilde{R}^2, C)$. This generalization is made simply by decomposing the boundary of a fuzzy simple region into two parts: the boundary of the boundary and the boundary of the interior. These 12 conditions can be easily proven based on Proposition 3.17. For the simplicity's sake, call the interior, the interior of the boundary and the exterior "*topoarea*", and call the boundary of the boundary "*topoline*". The 12 conditions are as follows (conditions (1) and (5) are unchanged):

(1) The exteriors of two fuzzy regions intersect each other;
(2) Any part of *A* intersects at least one part of *B*, and vice versa;
(3) If one topoarea of *A* intersects two topoareas of *B*, then this topoarea must also intersect with the topoline of *A*, and vice versa;
(4) If both topolines do not intersect each other, then one topoline must intersect at least its opposite exterior;
(5) If both topolines intersect the opposite interiors, then the topolines must also intersect each other;
(6) If the interior (or the interior of the boundary, respectively) of *A* intersects the exterior of *B*, then the topoline of *A* must also intersect the exterior of *B*, and vice versa;
(7) If both interiors (or both interiors of the boundary, respectively) are disjoint and the topoline of *A* intersects the interior (or the interior of the boundary, respectively) of *B*, then the two topolines must intersect each other, and vice versa;
(8) If the interior (or the interior of the boundary, respectively) of *A* is a subset of the closure of *B* (or the closure of the interior of *B* or the closure of the boundary of *B*, respectively), then the boundary must intersect the closure of *B* (or the closure of the interior of *B* or the closure of the boundary of *B*, respectively), and vice versa;
(9) If both interiors (or both interiors of the boundary, respectively) are disjoint, then the interior (or the interior of the boundary, respectively) of *A* intersects

either the interior of the boundary of *B* (or the interior of *B*, respectively) or the exterior of *B*, and vice versa;

(10) If the interior (or the interior of the boundary, respectively) of *A* does not intersect the closure of *B*, then the topoline of *A* must intersect the exterior of *B*, and vice versa;

(11) If the topoline of *A* intersects one topoarea of *B*, then at least one topoarea of *A* intersects this topoarea of *B*;

(12) If the closure of *A* is a subset of the interior (or the interior of the boundary, respectively) of *B*, then the exterior of *A* must intersect the interior (or the interior of the boundary, respectively) of *B*, and vice versa.

For the identification, we adopt the 44 relations identified by the 3*3-intersection approach at first since the boundary is the union of the interior of the boundary and the boundary of the boundary of a fuzzy set. Then we extend each of the relations according to the above geometric conditions. One hundred and fifty-two (152) relations are identified by using the 4*4-intersection approach (see Appendix 2).

3.8 Conclusions and discussions

This chapter proposed an approach for identifying topological relations between two simple fuzzy regions. The basic idea is to form a crisp fts for fuzzy sets. By recognizing mutually disjoint topological parts of fuzzy sets in cts, an extension is made from the 9-intersection to the 4*4-intersection and 5*5-intersection matrices. This extends the 9-intersection approach from the crisp domain into the fuzzy domain and unifies the conclusion that is derived by Egenhofer and Franzosa as well as Clementini and Di Felice. By using this approach, topological relations can be formally derived between two simple fuzzy regions in the crisp fts.

Different types of intersection matrices can be used in different situations. The 3*3-intersection can be used when the boundary is not distinguished in detail; the 4*4-intersection can be adopted when the disjointness between the boundary of the interior and the boundary of the closure of the fuzzy region cannot hold; and the 5*5-intersection approach can be applied when these two boundaries can be defined as separated closed sets.

This approach is also applicable for the identification of topological relations between fuzzy sets and crisp sets since the crisp fts can accommodate both fuzzy and crisp sets.

Several problems have to be further investigated. One is how to derive topological relations between two fuzzy sets in the pure fuzzy topological space. The crisp fts is too coarse since in such a space the membership values are totally neglected. For example, we cannot compare the membership values between two simple fuzzy regions. A simple fuzzy region is defined in the crisp fts. It is usually neither closed nor open in the crisp fts. This is inconsistent with the definition of a crisp simple region in cts where a crisp simple region is a regular closed set.

Another problem is how to build up a fuzzy model for GIS applications when fuzzy spatial objects are involved. In GIS applications, there could be fuzzy lines, fuzzy points and fuzzy regions. It is also necessary to identify the topological relations between them.

Chapter Four

Topological Relations

in a General Fuzzy Topological Space

4.1 Introduction

In the previous chapter, the 3*3, 4*4 and even 5*5-intersection matrices were presented based on the interior, the boundary and the exterior of a fuzzy set in the crisp fts, and a simple fuzzy region was defined on the crisp fts. The topological relations between two simple fuzzy regions were also identified in the crisp fuzzy Euclidean space.

It is noticed that all these realizations are based on the crisp fts. This kind of space is too coarse from the topological point of view. Let (X, C) be a crisp fts and (X, τ) be a cts. Then the topology C is equal to τ on X. The only difference is that in (X, C) fuzzy sets exist whereas (X, τ) has no fuzzy set. Therefore the interior, the boundary and the exterior of a fuzzy region are all crisp sets. We repeat Figure 3.10(II) in Figure 4.1. The simple fuzzy region *A* itself is not a closed set. The boundary of *A* does not belong to *A*, but it is projected to 1 of [0,1] (Figure 4.1).

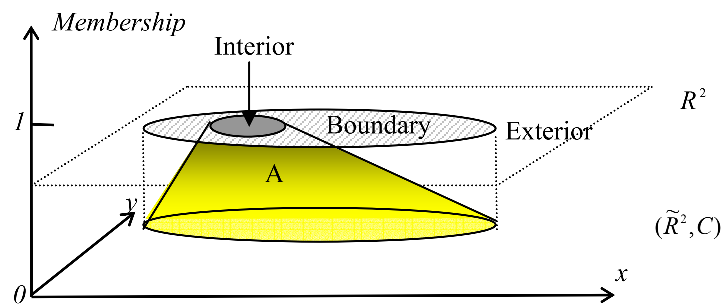


Figure 4.1 Problems of the definition of a simple fuzzy region in $(\widetilde{R}^2, C)$

The topological relations between two simple fuzzy regions are identified based on these crisp topological parts. The shortcomings of the definition of a simple fuzzy region are as follows. (1) The membership values of a fuzzy set are totally neglected. For example, we cannot compare the membership values between two fuzzy regions. (2) A simple fuzzy region could be neither closed nor open. This is inconsistent with the definition of a crisp simple region in the crisp topological space where a crisp simple region is a regular closed set.

In Section 3.3.2, it is pointed out that in order to identify the topological relations between two fuzzy sets, each subset of any fuzzy set in the intersections should be topological properties, and they are mutually disjoint. Proposition 3.4 has shown that the boundary is disjoint with the interior if and only if the interior of a fuzzy set is crisp. Therefore, unless we specify that the interior of a fuzzy set is crisp, it is impossible to identify the topological relations in the general topological space in terms of the interior, boundary and exterior of a fuzzy set.

The question is whether it is possible to find other topological properties of a fuzzy set that are mutually disjoint. This chapter will introduce a method to formalize and identify the topological relations between two simple fuzzy regions in a general fuzzy topological space. Section 4.2 proposes several topological properties, such as the fringe, the core, the outer, the internal fringe and the frontier of a fuzzy set. Section 4.3 introduces two 3*3-intersection matrices and one 4*4-intersection matrix based on these topological properties in the general fuzzy topological space. Section 4.4 defines a simple fuzzy region formally in a general fuzzy topological space. Section 4.5 discusses the topological relations between two simple fuzzy regions. Section 4.6 compares the different topological spaces and the definitions of a simple fuzzy region. Section 4.7 presents the conclusions and discussions.

4.2 More topological properties

4.2.1 Core and fringe

The interior, the boundary and the exterior might not be mutually disjoint in a general fuzzy topological space. However, it is possible to find other subsets that are mutually disjoint.

Definition 4.1 *Let* (X,δ) *be an fts. The subset of the closure of fuzzy set A where* $(A^- \cap A^{c-})(x) = 0$ *for all* $x \in X$ *is called the core of A in X and denoted by* $A^\oplus$. *The subset of the closure of fuzzy set A where* $(A^- \cap A^{c-})(x) > 0$ *for all* $x \in X$ *is called the fringe of A and denoted by* ℓA. *In other words,* $A^\oplus(x) = A^-(x)$ *iff* $(A^- \cap A^{c-})(x) = 0$. $\ell A(x) = A^-(x)$ *iff* $(A^- \cap A^{c-})(x) > 0$ *for all* $x \in X$.

For example, in an fts $(\widetilde{R}, d)$ such that *d* is induced from the usual topological space *R*, Let *A* be a fuzzy closed interval:

$$A = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = 0.3 & if \quad |x| = 2 \\ A(x) = \left|0.7 * |x| - 1.7\right| & if \quad 1 < |x| < 2 \\ A(x) = 1 & if \quad |x| \le 1 \end{cases}$$

then,

$$A^{o} = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = 0 & if \quad |x| = 2 \\ A(x) = \left|0.7 * |x| - 1.7\right| & if \quad 1 < |x| < 2 \\ A(x) = 1 - \varepsilon & if \quad |x| = 1 \\ A(x) = 1 & if \quad |x| < 1 \end{cases}, \quad \partial A = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = 0.3 & if \quad |x| = 2 \\ A(x) = \left|0.7 |x| - 1.7\right| & if \quad 1 < |x| < 2 \\ A(x) = 1 & if \quad |x| = 1 \\ A(x) = 0 & if \quad |x| < 1 \end{cases},$$

$$A^{\oplus} = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = 0 & if \quad |x| = 2 \\ A(x) = 0 & if \quad 1 < |x| < 2 \\ A(x) = 0 & if \quad |x| = 1 \\ A(x) = 1 & if \quad |x| < 1 \end{cases}, \quad \ell A = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = 0.3 & if \quad |x| = 2 \\ A(x) = \left|0.7 |x| - 1.7\right| & if \quad 1 < |x| < 2 \\ A(x) = 1 & if \quad |x| = 1 \\ A(x) = 0 & if \quad |x| < 1 \end{cases}.$$

The interior, the boundary, the core and the fringe are illustrated in Figure 4.2.

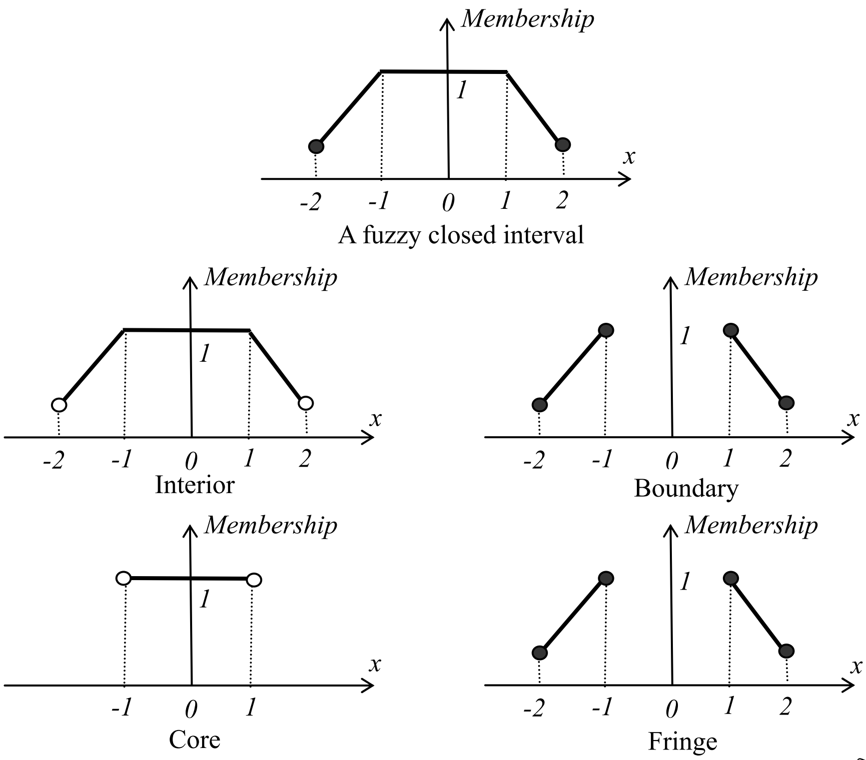


Figure 4.2 Interior, boundary, core, and fringe of a fuzzy closed interval in $(\widetilde{R}, d)$

Obviously the following proposition holds for a subset in the fts.
Proposition 4.2 *Let A be a fuzzy set in fts* (X,δ) *. Then* $A^{-} = A^{\oplus} \cup \partial A = A^{\oplus} \cup \ell A$.

Denote the support of the closure of fuzzy set *A* by A^{+} . The complement of A^{+} is called the *outer* of fuzzy set *A* and is written as $A^{=}$. Obviously both A^{+} and $A^{=}$ are crisp. The following proposition obviously holds.

Proposition 4.3 *Let A be a fuzzy set in fts* (X,δ) *. Then* $A^{\oplus}$, ℓA *and* $A^{=}$ *are mutually disjoint.*

It is important to investigate the relationships between the fringe and the core with the boundary and the interior of a fuzzy set in the fts.

Proposition 4.4 *Let A be a fuzzy set in fts* (X,δ) *. Then* $A^{\oplus}$ is the only crisp subset of A^{o} .
Proof: According to the definition of the core, $\forall x \in X$, $A^{\oplus}(x) = A^{-}(x) > 0$ iff $(A^{-} \cap A^{c-})(x) = 0$. Then either $A^{-}(x) = 0$ or $A^{c-}(x) = 0$. (i) When $A^{-}(x) = 0$, then $A^{\oplus}(x) = 0$. Contradiction. (ii) When $A^{c-}(x) = 0$, then $A^{oc}(x) = 0$ according to Proposition 2.9, so $A^{o}(x) = A^{-}(x) = 1$. It shows $\forall x \in X$, if $A^{\oplus}(x) > 0$, then $A^{\oplus}(x) = 1$ and $A^{o}(x) = 1$. On the other hand, if $A^{o}(x) = 1$, then $A^{oc}(x) = 0$, so $A^{c-}(x) = 0$. Therefore $(A^{-} \cap A^{c-})(x) = 0$. It follows that if $A^{o}(x) = 1$, then $A^{\oplus}(x) = 1$.
Proposition 4.4 shows that $A^{\oplus}$ is crisp and $A^{\oplus} \subseteq A^{o}$.

Proposition 4.5 *Let A be a fuzzy set in fts* (X,δ) .

(1) $A^{\oplus\oplus} \subseteq A^{\oplus}$;
(2) $A^{o\oplus} = A^{\oplus}$, $A^{-\oplus} = A^{\oplus}$;
(3) *If* $A \supseteq B$ *, then* $A^{\oplus} \supseteq B^{\oplus}$;
(4) $A^{\oplus} \cap B^{\oplus} = (A \cap B)^{\oplus}$;
(5) *If* $A^{\oplus}$ *is open, then* $A^{\oplus} \cap \ell(A^{\oplus}) = \varnothing$;
(6) $A^{\oplus\oplus} = A^{\oplus}$ *iff* $A^{\oplus}$ *is open.*

Proof:

(1) $\forall x \in X$, if $A^{\oplus\oplus}(x) > 0$, then $(\ell(A^{\oplus}))(x) = 0$ according to Proposition 4.3. Then $(A^{\oplus -} \cap A^{\oplus c-})(x) = 0$. Since $A^{\oplus -}(x) > 0$, we have $A^{\oplus c-}(x) = 0$, and $A^{\oplus}(x) = 1$. Therefore $A^{\oplus\oplus} \subseteq A^{\oplus}$.
(2) $A^{o\oplus}$ is the subset of A^{o-} where $(A^{o-} \cap A^{oc-})(x) = 0$ and $A^{o-}(x) > 0$, $\forall x \in X$. It follows that $A^{oc-}(x) = 0$ and $A^{o}(x) = 1$. Then $A^{\oplus}(x) = A^{o\oplus}(x) = 1$. Therefore $A^{o\oplus} = A^{\oplus}$. Similarly, $A^{-\oplus}$ is the subset of

A^- where $(A^- \cap A^{-c-})(x) = 0$. Since $(A^- \cap A^{-c-})(x) \leq (A^- \cap A^{c-})(x)$, so $\forall x \in X$, when $(A^- \cap A^{c-})(x) = 0$, then $(A^- \cap A^{-c-})(x) = 0$. Therefore $A^{-\oplus} = A^{\oplus}$.

(3) If $A \supseteq B$, then $A^o \supseteq B^o$. Since $A^{\oplus}$ and $B^{\oplus}$ are the crisp subsets of A^o and B^o, it follows $A^{\oplus} \supseteq B^{\oplus}$.

(4) Since $A^{\oplus}$ is the only crisp subset of A^o, and $B^{\oplus}$ is the only crisp subset of B^o, then $A^{\oplus} \cap B^{\oplus}$ is the crisp subset of $A^o \cap B^o$. Since $A^o \cap B^o = (A \cap B)^o$, $A^{\oplus} \cap B^{\oplus}$ is also the crisp subset of $(A \cap B)^o$. Therefore, $A^{\oplus} \cap B^{\oplus} = (A \cap B)^{\oplus}$.

(5) $\forall x \in X$, $l(A^{\oplus})(x) = A^{\oplus -}(x)$ iff $(A^{\oplus -} \cap A^{\oplus c-})(x) > 0$. If $A^{\oplus}$ is open, then $A^{\oplus c-} = A^{\oplus c}$. Thus, $\forall x \in X$, when $l(A^{\oplus})(x) > 0$, then $A^{\oplus -}(x) > 0$ and $A^{\oplus c-}(x) = A^{\oplus c}(x) > 0$. According to Proposition 4.4, $A^{\oplus}$ is crisp. Then $A^{\oplus c}$ is crisp and $A^{\oplus c}(x) = 1$. Therefore, if $\forall x \in X$, $l(A^{\oplus})(x) > 0$, then $(A^{\oplus} \cap l(A^{\oplus}))(x) = 0$.

(6) If $A^{\oplus}$ is open, then $A^{\oplus} = A^{\oplus o}$. According to Proposition 4.2, $A^{\oplus -} = A^{\oplus\oplus} \cup \ell(A^{\oplus}) = A^{\oplus o} \cup \ell(A^{\oplus}) = A^{\oplus} \cup \ell(A^{\oplus})$. Since $A^{\oplus\oplus} \subseteq A^{\oplus o} \subseteq A^{\oplus}$, therefore $A^{\oplus\oplus} = A^{\oplus o} = A^{\oplus}$.

Since $A^{\oplus\oplus} \subseteq A^{\oplus o}$ and $A^{\oplus o} \subseteq A^{\oplus}$, therefore $A^{\oplus\oplus} = A^{\oplus} = A^{\oplus o}$. $A^{\oplus}$ is open.

Proposition 4.6 *Let A be a fuzzy set in fts* (X, δ).

(1) $\ell A \subseteq \partial A$;

(2) $\ell A = \partial A$ *iff* ℓA *is closed*;

(3) $(\ell A)^- = \partial A$.

Proof: These are obvious.

Proposition 4.7 *Let A be a fuzzy set in fts* (X, δ).

(1) $\ell A = A^- \cap A^{\oplus c}$;

(2) $(\ell A)^{\oplus} = \varnothing$;

(3) $\ell(\ell A) = \partial A$;

(4) *If* $A^{\oplus}$ *is open, then* $(\ell(A^{\oplus}) \subseteq \ell A$.

Proof:

(1) Since $A^{\oplus} \cap \ell A = \varnothing$, we have $A^{\oplus c} \supseteq \ell A$, and $\ell A \subseteq A^- \cap A^{\oplus c}$. On the other hand, $\forall x \in X$, if $\ell A(x) > 0$, then $\ell A(x) = A^-(x) \supseteq (A^- \cap A^{\oplus c})(x)$. Therefore, $\ell A = A^- \cap A^{\oplus c}$.

(2) $(\ell A)^{\oplus} = (A^- \cap A^{\oplus c})^{\oplus} = A^{-\oplus} \cap A^{\oplus c\oplus} = A^{\oplus} \cap A^{\oplus c\oplus} \subseteq A^{\oplus} \cap A^{\oplus c} = \varnothing$. The second equation is based on Proposition 4.5(4). The third equation is based on Proposition 4.5(2).

(3) $\ell(\ell A) = (\ell A)^- \cap (\ell A)^{\oplus c} = (\ell A)^- = \partial A$.

(4) If $A^{\oplus}$ is open, then $\ell(A^{\oplus}) = A^{\oplus -} \cap A^{\oplus\oplus c} = A^{\oplus -} \cap A^{\oplus c} \subseteq A^{-} \cap A^{\oplus c} = \ell A$.

According to Propositions 4.7(3) and 4.6(3), if the fringe of a fuzzy set is closed, then the fringe of the fringe is equal to the fringe, which is the boundary of a fuzzy set.

Proposition 4.8 *Let A be a fuzzy set in fts* (X,δ) . $\partial A \cap A^{\oplus} = \varnothing$ *iff* $\partial A = \ell A$.

Proof:

$\Leftarrow$ This is obvious since $A^{\oplus} \cap \ell A = \varnothing$.

$\Rightarrow$ If $\partial A \cap A^{\oplus} = \varnothing$ then $\partial A \subseteq A^{\oplus c}$. Then $\ell A = A^{-} \cap A^{\oplus c} \supseteq A^{-} \cap \partial A = \partial A$. On the other hand, according to proposition 4.6(1) $\partial A \supseteq \ell A$, therefore $\partial A = \ell A$.

Proposition 4.9 *Let A be a fuzzy set in fts* (X,δ) *. If* $A^{\oplus}$ *is open, then* $\partial A = \ell A$.

Proof: If $A^{\oplus}$ is open, then $\ell A = A^{-} \cap A^{\oplus c}$ is closed. Then $\partial A = \ell A$.

According to these propositions, it can be perceived that the core might not be open. It is just the crisp subset of the interior of a fuzzy subset. Similarly, the fringe might not be closed. The fringe can also be regarded as another definition of the fuzzy boundary of a fuzzy set. The fringe is a subset of the boundary of a fuzzy set. Proposition 4.9 shows that the fringe of a fuzzy set is closed and is equal to the boundary if the core is open. In Figure 4.2, the boundary is equal to the fringe of fuzzy interval A in $(\tilde{R},d)$ since the fringe is closed.

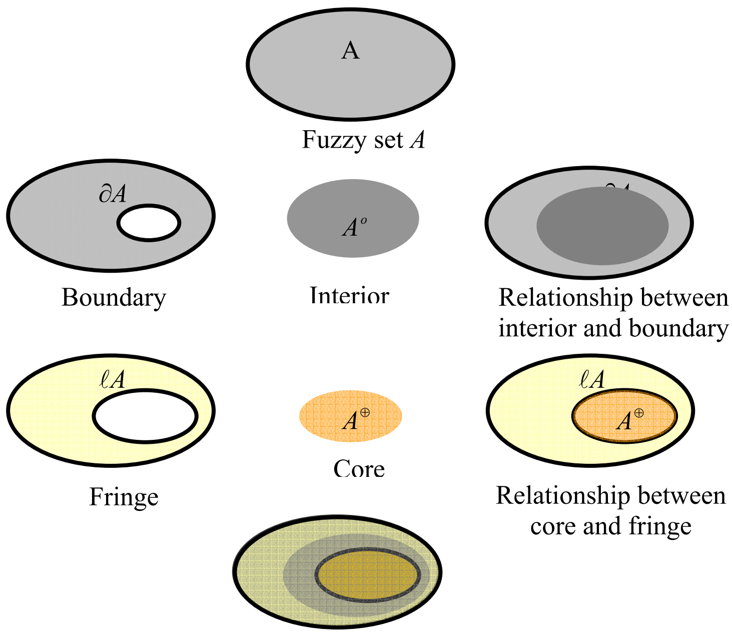


Figure 4.3 Relationships between core, interior, fringe and boundary of a fuzzy set in fuzzy topological space

The above propositions reveal the relationships between the core and the interior, and the fringe and the boundary of a fuzzy set *A* in a general fts. The relationships are illustrated in Figure 4.3. In Figure 4.3, a fuzzy set is represented by a modified VENN diagram. A VENN diagram is modified according a fuzzy set in the fuzzy Euclidean space. For example, since the boundary of a fuzzy set is usually located at the rim of a fuzzy set in the fuzzy Euclidean space, the boundary is then represented by an annulus; the interior is drawn by an open disk since usually it has no boundary.

Now we can check if the core, fringe and outer are topological properties.
Proposition 4.10 *Let A be a fuzzy set in fts* (X,δ) *. The core, the fringe and the outer of fuzzy set A are topological properties.*
Proof: The core and the outer are topological invariants since any (fuzzy) homeomorphic mapping is crisp-subset preserving. Let $f^{\rightarrow}$ be a homeomorphic mapping from fts (X,δ) to fts (Y,θ). For every $A \subseteq X$, there is $B \subseteq Y$ such that $f^{\rightarrow}(A) = B$. Since any homeomorphic mapping is bijective, *i.e.*, $f^{\rightarrow}(\ell A) = f^{\rightarrow}(A^{-} \cap A^{\oplus c}) = f^{\rightarrow}(A^{-}) \cap f^{\rightarrow}(A^{\oplus c}) = B^{-} \cap B^{\oplus c} = \ell B$, therefore the fringe is also a topological invariant.

Proposition 4.10 shows that not only are the interior, the boundary, the closure and the exterior of a fuzzy set topological properties in the fts, but also the core, the fringe and the outer of a fuzzy set hold such properties.

4.2.2 Internal and frontier

Besides the core and the fringe, it is possible to derive other topological properties of a fuzzy set in the fts.

Definition 4.11 *Let* (X,δ) *be an fts. The subset of the closure of fuzzy set A where* $A^{-}(x) > A^{o}(x)$ *for all* $x \in X$ *is called the frontier of A in X and denoted by* $\ell^{e} A$. *The subset of the closure of fuzzy set A where* $A^{-}(x) = A^{o}(x)$ *for all* $x \in X$ *is called the internal of A and denoted by* A^{i} .

In other words, $A^{i}(x) = A^{-}(x)$ iff $A^{-}(x) = A^{o}(x)$. $\ell^{e} A(x) = A^{-}(x)$ iff $A^{-}(x) > A^{o}(x)$ for all $x \in X$. The concept of frontier can also refer to the definition of boundary (Wu and Zheng 1991).

The internal and the frontier of a closed interval *A* that is described in Figure 4.2 are:

$$A^{i} = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = 0 & if \quad |x| = 2 \\ A(x) = \left|0.7 * |x| - 1.7\right| & if \quad 1 < |x| < 2 \\ A(x) = 0 & if \quad |x| = 1 \\ A(x) = 1 & if \quad |x| < 1 \end{cases}, \quad \ell^{e} A = \begin{cases} A(x) = 0 & if \quad |x| > 2 \\ A(x) = \varepsilon & if \quad |x| = 2 \\ A(x) = 0 & if \quad 1 < |x| < 2 \\ A(x) = 1 - \varepsilon & if \quad |x| = 1 \\ A(x) = 0 & if \quad |x| < 1 \end{cases}.$$

They are illustrated in Figure 4.4.

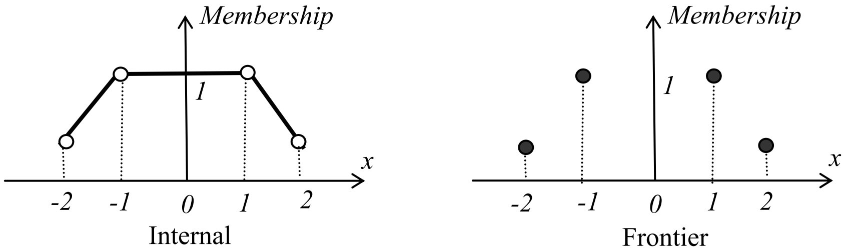


Figure 4.4 Internal and frontier of a fuzzy closed interval in $(\widetilde{R}, d)$

The following proposition is obvious.
Proposition 4.12 *Let A be a fuzzy set in fts* (X, δ).

(1) $\ell^e A$ *and* A^i *are disjoint with each other*, $\ell^e A \cap A^i = \varnothing$;
(2) $A^- = \ell^e A \cup A^i$.

Proposition 4.13 *Let A and* $\ell^e A$ *be closed sets in fts* (X, δ). *Then* $\ell^e A = \ell^e(\ell^e A)$.
Proof: Since A is closed, then $A \supseteq \ell^e A$, therefore, $\ell^e A \supseteq \ell^e(\ell^e A)$. We will verify the other inequality, that is, $\ell^e A \subseteq \ell^e(\ell^e A)$. Let B be a subset of $(\ell^e(\ell^e A))$, such that for all $\forall x \in X$ with $(\ell^e A)(x) \geq (\ell^e A)^o(x)$, then $B(x) = (\ell^e A)(x)$. Let us prove that B also satisfies "$\forall x \in X$ with $A^-(x) > A^o(x)$, then $B(x) \geq A(x)$". In fact, if $y \in X$ satisfies $A^-(y) > A^o(y)$, then $(l^e A)(y) = A^-(y)$. Then, when $A^-(y) > A^o(y)$, there is $(\ell^e A)(y) > A^o(y)$. Since A is closed, then $\ell^e A \subseteq A$, and $(\ell^e A)^o \subseteq A^o$, thus $(\ell^e A)(y) > (\ell^e A)^o(y)$. Therefore $B(y) \geq (\ell^e A)(y) = A^-(y)$. Since y is arbitrarily chosen, therefore $B \supseteq A$.

Proposition 4.14 *Let A be a fuzzy set in fts* (X, δ). *The internal and the frontier of fuzzy set A are topological properties.*
Proof: Let $f^{\to}$ be a homeomorphic mapping from fts (X, δ) to fts (Y, θ). For every $A \subseteq X$, there is a $B \subseteq Y$ such that $f^{\to}(A) = B$. Since $f^{\to}(A^-) = B^-$, and $f^{\to}(A^o) = B^o$, then for every $x \in X$, if $A^-(x) > A^o(x)$, then there is a $y \in Y$ such that $B^-(y) > B^o(y)$. Similarly, for every $x \in X$, if $A^-(x) = A^o(x)$, then there is a $y \in Y$ such that $B^-(y) = B^o(y)$. Therefore for every $x \in X$, if $A^-(x) > A^o(x)$, then $(\ell^e A)(x) = A^-(x)$, there is a $y \in Y$ such that $B^-(y) > B^o(y)$, that is, $(\ell^e B)(y) = B^-(y)$. Therefore $f^{\to}(\ell^e A) = \ell^e B$. So the frontier is a topological property. Similarly, the internal is also a topological property.

Now we investigate some relationships between the core, the internal and the interior, as well as the boundary, the fringe and the frontier of a fuzzy set in the fts.

Proposition 4.15 *Let A be a fuzzy set in fts* (X,δ).

(1) $A^{\oplus} \subseteq A^{i} \subseteq A^{o}$;

(2) $\partial A \supseteq \ell A \supseteq \ell^{e} A$.

Proof:

(1) According to Proposition 4.4, $A^{\oplus}$ is the crisp subset of A^{o}. According to the definition of A^{i}, it is a subset of A^{o}, and it contains the crisp subset of A^{o}.

(2) If $A^{-}(x) > A^{o}(x)$ for all $x \in X$, then $A^{o}(x) < 1$ and $A^{c-} = A^{oc}(x) > 0$. Therefore $(A^{-} \cap A^{c-})(x) = A^{-}(x) \wedge A^{c-}(x) > 0$, therefore $\ell A(x) > 0$. So $\ell A \supseteq \ell^{e} A$

According to Proposition 4.15(2), the frontier is contained in the fringe of a fuzzy set. This is the reason why we adopt the symbol $\ell^{e} A$ for the frontier of a fuzzy set *A*. The frontier can be regarded as another definition of the fuzzy boundary of a fuzzy set in the fts. The difference between the frontier and boundary III is that boundary III is closed but the frontier could be not closed. If the frontier is closed, then it is equal to boundary III. Since $\ell^{i} A \subseteq \ell A$, the frontier is finer than the fringe and the boundary of a fuzzy set in the fts. In the example of Figure 4.4, boundary III is the frontier of interval *A* in $(\widetilde{R}, d)$ since the frontier is a closed set.

Figure 4.5 shows the above proposition in a VENN diagram. The relationships between the core, the internal and the interior of a fuzzy set *A* are drawn in Figure 4.5(1). That is, the core is the subset of the internal, which is the subset of the interior of a fuzzy set *A*. The relationships between the frontier, the fringe, and the boundary of a fuzzy set *A* are shown in Figure 4.5(2).

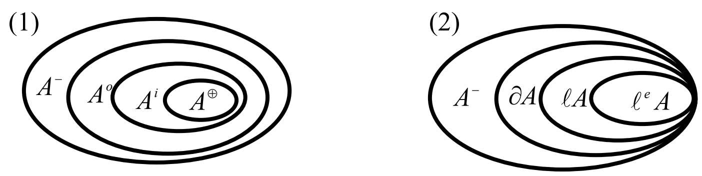


Figure 4.5 (1) Relationships between core, internal and interior; (2) Relationships between frontier, fringe and boundary of a fuzzy set in fuzzy topological space

The relationships between the core, the interior, the fringe and the boundary of a fuzzy set *A* are illustrated in Figure 4.6 by a modified VENN diagram. In Figure 4.6, the boundary and the interior are copied from Figure 4.3.

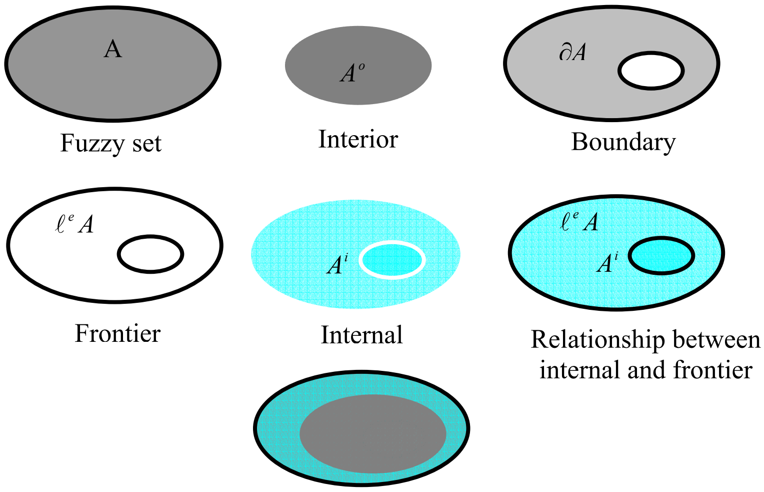


Figure 4.6 Relationships between internal, interior, frontier and boundary of a fuzzy set in fuzzy topological space

The relationships between the core, the internal, the interior, the frontier, the fringe, and the boundary of a fuzzy set A are illustrated in Figure 4.7. Figure 4.7 is derived by merging the relationships in Figure 4.3 and Figure 4.6.

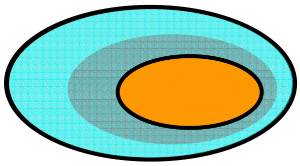


Figure 4.7 Relationships between core, inner, interior, frontier, fringe and boundary of a fuzzy set in fuzzy topological space

4.2.3 Internal of fringe

We denote the subset of the internal of a fuzzy set A where $A^i \cap A^{\oplus c}$ by $\ell^i A$. Then the following proposition is obvious.

Proposition 4.16 *Let A be a fuzzy set in fts* (X, δ).

(1) $\ell A = \ell^e A \cup \ell^i A$;

(2) $A^i = A^{\oplus} \cup \ell^i A$.

Proposition 4.16 shows $\ell^i A \subseteq \ell A$. Therefore we can call $\ell^i A$ the *internal of the fringe* of a fuzzy set A (or *the internal fringe* for short) in the fts. Figure 4.8 shows the internal fringe of the closed interval A that is described in Figure 4.2.

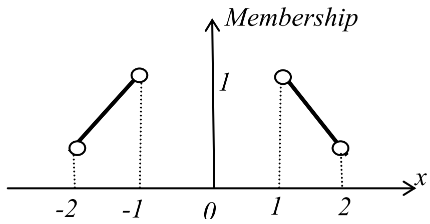


Figure 4.8 Internal fringe of a fuzzy closed interval in $(\widetilde{R}, d)$

The following proposition holds according to the definitions.

Proposition 4.17 *Let A be a fuzzy set in fts* (X, δ). $A^{=}, \ell^e A, \ell^i A, A^{\oplus}$ *are mutually disjoint, and they are topological properties.*

Figure 4.9 shows that the core, the internal fringe and the frontier of fuzzy set *A* are mutually disjoint in the fts by a modified VENN diagram.

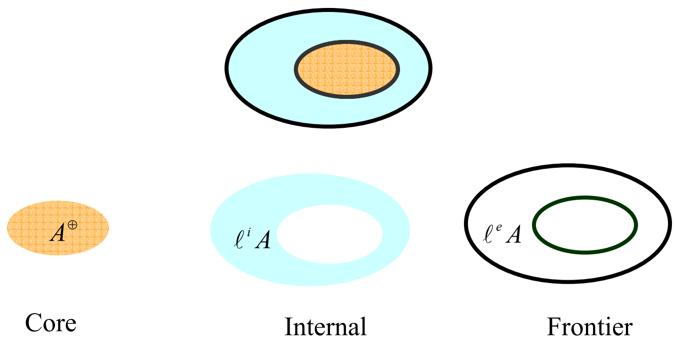


Figure 4.9 Core, internal fringe and frontier of a fuzzy set are mutually disjoint in fuzzy topological space

Surely, we can also decompose the boundary of a fuzzy set into two other disjoint subsets. Define *the internal of the boundary* $\partial^i A$ of fuzzy set *A* as the subset of the boundary where $A^-(x) = A^o(x)$ for all $x \in X$. Call the internal of the boundary *the internal boundary* for simplicity. For the duality, we denote *the frontier of the boundary* of fuzzy set *A* by $\partial^e A$. The union of the frontier of the boundary and the internal boundary will be the boundary of fuzzy set *A*. The internal fringe and the internal boundary will be equal when the fringe of a fuzzy set is closed. Since the boundary is not disjoint with the core, maybe the internal boundary is not disjoint with the core. The definition will not benefit either the definition of fuzzy spatial objects or the intersection matrix. We focus more on the frontier and the internal fringe of a fuzzy set.

Proposition 4.18 *Let A be a crisp subset in fts* (X, δ). $\ell^i A = \varnothing$.

Proof: If $\exists x \in X$, $A^o(x) = 1$, then $A^-(x) = A^o(x) = 1$. Then, $(A^- \cap A^{c-})(x) =$

$(A^- \cap A^{oc})(x) = 0$, then $\ell A(x) = 0$. Therefore $\forall x \in X$, iff $A^o(x) < 1$, then $\ell A(x) > 0$. When $A^o(x) < 1$, then $A^-(x) \geq A(x) = 1$. Then $A^-(x) > A^o(x)$. Then $\ell^i A(x) = 0$. Therefore $\ell^i A = \varnothing$.

Corollary 4.19 *Let A be a fuzzy set in fts* (X, δ) . *Then* $\ell^i(A^\oplus) = \varnothing$

Proposition 4.20 *Let A be a fuzzy set in fts* (X, δ) . *If* $A^\oplus$ *is open, then* $\ell^e(A^\oplus) \subseteq \ell^e A$.

Proof: According to Proposition 4.7(4), if $A^\oplus$ is open, then $\ell(A^\oplus) \subseteq \ell A$. Therefore $\ell^e(A^\oplus) \subseteq \ell A$. For all $x \in X$, if $\ell^e(A^\oplus)(x) > 0$, then $A^{\oplus -}(x) > A^{\oplus o}(x)$. Since $A^-(x) \geq A^{\oplus -}(x)$, and $A^{\oplus o}(x) \leq A^o(x)$, therefore $A^-(x) > A^o(x)$. That is, for all $x \in X$, when $A^{\oplus -}(x) > A^{\oplus o}(x)$, then $A^-(x) > A^o(x)$. Therefore $\ell^e(A^\oplus) \subseteq \ell^e A$.

4.3 Intersection matrices in general fuzzy topological space

The above section reveals the following facts:

(1) The closure of a fuzzy set can be decomposed into the core, the fringe and the outer, which are topological properties and mutually disjoint;
(2) The closure of a fuzzy set can be decomposed into the internal, the frontier and the outer, which are topological properties and mutually disjoint;
(3) The closure of a fuzzy set can be decomposed into the core, the internal fringe, the frontier and the outer, which are topological properties and mutually disjoint.

We have two possibilities of formalizing the 3*3-intersection matrix:

3*3-intersection matrix based on the core, the fringe and the outer: *Let A, B be two fuzzy sets in fts* (X, δ) .

$$I_{3*3} = \begin{bmatrix} A^\oplus \cap B^\oplus & A^\oplus \cap \ell B & A^\oplus \cap B^= \\ \ell A \cap B^\oplus & \ell A \cap \ell B & \ell A \cap B^= \\ A^= \cap B^\oplus & A^= \cap \ell B & A^= \cap B^= \end{bmatrix} \tag{1}$$

3*3-intersection matrix based on the internal, the frontier and the outer: *Let A, B be two fuzzy sets in fts* (X, δ) .

$$I_{3*3} = \begin{bmatrix} A^i \cap B^i & A^i \cap \ell^e B & A^i \cap B^= \\ \ell^e A \cap B^i & \ell^e A \cap \ell B & \ell^e A \cap B^= \\ A^= \cap B^i & A^= \cap \ell^e B & A^= \cap B^= \end{bmatrix} \tag{1'}$$

We can also formalize the 4*4-intersection matrix between two fuzzy sets in the fts.

4*4-intersection matrix: *Let A, B be two fuzzy sets in fts* (X,δ).

$$I_{4*4}=\begin{bmatrix} A^{\oplus}\cap B^{\oplus} & A^{\oplus}\cap \ell^{e}B & A^{\oplus}\cap \ell^{i}B & A^{\oplus}\cap B^{=} \\ \ell^{e}A\cap B^{\oplus} & \ell^{e}A\cap \ell^{e}B & \ell^{e}A\cap \ell^{i}B & \ell^{e}A\cap B^{=} \\ \ell^{i}A\cap B^{\oplus} & \ell^{i}A\cap \ell^{e}B & \ell^{i}A\cap \ell^{i}B & \ell^{i}A\cap B^{=} \\ A^{=}\cap B^{\oplus} & A^{=}\cap \ell^{e}B & A^{=}\cap \ell^{i}B & A^{=}\cap B^{=} \end{bmatrix} \tag{2}$$

The topological relations between two fuzzy sets can be derived by the intersection matrices (1), (1') and (2) based on the topological invariants in these intersections.

4.4 Formal definition of simple fuzzy region

In Chapter 3 we defined a simple fuzzy region in the crisp fts. Although it can represent the fuzzy spatial objects in GIS, the shortcomings are also obvious as discussed in Section 4.1. It is important to define a simple fuzzy region in a more general fuzzy topological space, so that it can be applied in any GIS application.

4.4.1 Formal definition of simple fuzzy region in a general fts

A simple crisp region, formally defined in the cts, is abstracted based on the topological properties. In the fts, more topological properties discussed in Section 4.2 can be adopted for the definition of a simple fuzzy region in the fts. The practical requirement decides which topological properties should be adopted. Since a simple fuzzy region is an extension of a simple crisp region from the cts to the fts, we adopt the following principles for the definition of a simple fuzzy region.

(1) A crisp subset of a simple fuzzy region should expose the same behaviors as a simple crisp region in the cts; and
(2) A simple crisp region in the fts should have the same behaviors as a simple crisp region in the cts.

A formal definition of a simple fuzzy region in the fts is given below:

Definition 4.21 *A fuzzy set is called a simple fuzzy region in a connected fts if it meets the following conditions:*

(1) *It is a non-empty proper double-connected closed set;*
(2) *The interior, the core and the outer are double-connected regular open;*
(3) *The support is equal to the support of the closure of the interior;*
(4) *The fringe is double-connected and the internal fringe is a double-connected open set;*
(5) *The frontier is a non-empty closed set.*

4.4.2 Explanations of definition

(1) Double-connectedness
A crisp simple region is connected in the cts. This means it cannot be the union of two non-empty disjoint open sets or two non-empty disjoint closed sets. In the fts, the connectedness of a fuzzy set can be basically extended into double-connectedness, both open-connectedness and closed-connectedness. A simple fuzzy region should be double-connected so that it can not be composed of two disjoint open sets, or two disjoint closed sets. That is, it should be in "one piece" in the topological sense.

The interior, the core, the fringe and the outer should be double-connected. The double-connectedness of the interior, the core, the fringe and the outer is a natural extension of a simple crisp region from the cts to the fts. That is, they should also be in one piece.

The reason that the core should be double-connected is as follows. According to our principle, the interior of a simple crisp region in the cts should be the interior of a simple crisp region in the fts. Since the interior of a simple crisp region in the cts is crisp, the interior of a simple crisp region should be also crisp in the fts. According to Proposition 4.4, the interior of a simple crisp region is equal to the core of a simple crisp region in the fts. Therefore, the core is required to be double-connected.

The internal fringe should be doubled-connected. It is an extra condition since it is not expected that the internal fringe will be separated into "several pieces" spatially.

(2) Open sets
The interior, the core and the outer are required to be regular open sets. That the interior should be regular open is the natural extension of the definition of a simple crisp region from the cts to the fts. The core should be regular open, since the interior of a simple crisp region in the cts should be equal to the core of a simple crisp region in the fts; and a simple fuzzy region should have the same "crisp interior" as a simple crisp region in the cts. A regular open core will further remove some fuzzy points whose membership values are less than 1 within the "area" of the core. The outer should also be regular open. In the cts, the exterior is automatically regular open since a simple crisp region is regular closed. The outer of a fuzzy set in the fts is similar to the exterior in the cts. Therefore, it is also natural to require it to be a regular open set.

The internal fringe should be open. It is an extra condition since it is expected that the internal fringe will be an "area" spatially.

(3) Closed sets
The frontier is defined as closed. According to Proposition 4.20, the frontier is a subset of the fringe, and it contains the fringe of the core. So spatially it performs like the boundary of the boundary of a crisp simple region in the cts (where the boundary of a simple crisp region in cts is equal to the boundary of the boundary). Therefore, it is required to be closed.

Another closed set is the simple fuzzy region itself. It should be closed, which is similar to the condition of a simple crisp region in cts. However, it is not required to be regular closed. The interior of a simple fuzzy region is required to be regular open. A regular open set will eliminate some irregular points and lines, similar to a regular closed set. For example, an open disk in crisp Euclidean space without a crisp point at its center is an open set but it is not regular since the interior of the closure will be the open disk (Figure 4.10). However, it will be seen that a simple fuzzy region may be not regular closed. In order to eliminate some "irregular" points, the simple fuzzy region should meet condition (3) of the definition.

Figure 4.10 An open set and a regular open set in R^2

4.4.3 Discussions on definition

Defining a simple fuzzy region raises the following arguments:

(1) Why not define a simple fuzzy region based on the interior, the boundary and the exterior?
A simple fuzzy region is defined based on the topological properties of the core, the internal fringe, the frontier and the outer of a fuzzy set in the fts. If a simple fuzzy region is defined just based on the interior, the boundary and the exterior, it could be "abnormal".

For example, we can define a metric *d* on the Euclidean plane R^2, such that for two points $x(a,b) \in R^2, y(c,d) \in R^2$, and $r(x,y) = \sqrt{(a-c)^2 + (b-d)^2}$:

$$d = \begin{cases} r & if \quad r < 1 \\ 1 & if \quad 1 \le r \le 2 \\ r-1 & if \quad r > 2 \end{cases}$$

Let (R^2, τ_d) be the metric topology on R^2. Define a fuzzy topology $(\widetilde{R}^2, \delta)$ on $\widetilde{R}^2$ such that δ is the induced topology from R^2. Let *A* be a fuzzy closed disk and it is the upper semicontinuous mapping from R^2 to [0,1]. For example, let *A* be a closed fuzzy disk around point *x*, such that $A_d(x) = \{y \in R : d(x,y) \le 2\}$ and its membership values are ($\varepsilon > 0$):

$$\mu_A = \begin{cases} 1 & if \quad 0 \le d(x,y) \le 1 \\ 2 - d(x,y) & if \quad 1 < d(x,y) < 2 \\ \varepsilon & if \quad d(x,y) = 2 \\ 0 & if \quad d(x,y) > 2 \end{cases}$$

Figure 4.8 illustrates the closed disk *A*. It is a closed set. The boundary of *A* on $(\widetilde{R}^2, \delta)$ is ∂A:

$$\mu_{\partial A}(y) = \begin{cases} 0 & if \quad 0 \le d(x,y) < 1 \\ 2 - d(x,y) & if \quad 1 \le d(x,y) < 2 \\ \varepsilon & if \quad d(x,y) = 2 \\ 0 & if \quad d(x,y) > 2 \end{cases}$$

Then on the boundary, there is an area $B(x) = \{y \in R : 1 \le r(x,y) \le 2\}$ whose membership value is 1. This area should be treated as the core in GIS applications. In Figure 4.11 *B* is marked in red.

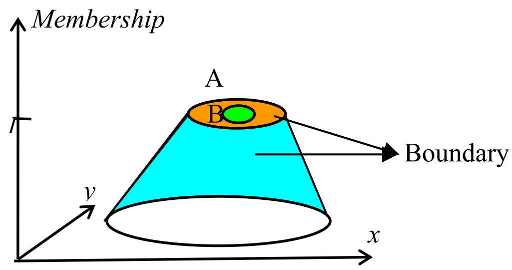


Figure 4.11 An abnormal simple fuzzy region

(2) Why not define a simple fuzzy region based on the internal, the frontier and the outer?
It is possible to define a simple fuzzy region based on the internal, the frontier and the outer. For example, we can change condition (2) in Definition 4.21 into a condition such that the internal is composed of two non-empty disjoint double-connected open sets. However, since the core is not required to be open, the crisp subset of a fuzzy set may not be open.

(3) Why not define a simple fuzzy region based on other topological properties?
In the fts, apart from these topological properties, we can identify more topological properties. For example, we can define the subset of the closure of a fuzzy set in the fts in such a way that the interior intersects with the interior of the complement of the set. It is also a topological property. However, this subset has no direct relationship with fuzzy boundary. As we know, the most important aspect for GIS is to model the fuzzy boundary of a fuzzy spatial object. Therefore, it is better to neglect these topological properties for GIS.

4.4.4 Properties of a simple fuzzy region

Some properties of a simple fuzzy region are proven in the following propositions.

Proposition 4.22 *Let A be a simple fuzzy region in a connected fts X.*

(1) *The boundary of A is equal to the fringe of A: $\partial A = \ell A$, and they are not empty;*
(2) *The internal boundary is equal to the internal fringe of A: $\partial^i A = \ell^i A$;*

(3) *The frontier of the boundary is equal to the frontier of A:* $\partial^e A = \ell^e A$ *, and they are not empty;*
(4) *The fringe of the outer of A is equal to the boundary of the outer of A and they are not empty:* $\ell(A^{=}) = \partial(A^{=}) \neq \varnothing$.

Proof:
(1) Since the core of simple fuzzy region *A* is open, the fringe is closed, which is equal to the boundary. They are not empty. Suppose $\partial A = \varnothing$, then $X = A^{\oplus} \cup A^{=}$. Then *X* is not connected. Therefore, $\partial A \neq \varnothing$.
(2) It is obvious according to (1).
(3) The first part is obvious since the fringe is equal to the boundary. Suppose $\partial^e A = \varnothing$, then $A = A^{\oplus} \cup \partial^i A$. Then *A* is not open-connected. Therefore, $\partial A \neq \varnothing$.
(4) The first part is because the outer is open. The second part is because *X* is connected.

Proposition 4.22(1) shows that, for a simple fuzzy region, the fringe is equal to the boundary. For simplification, we will write the fringe ℓA of a simple fuzzy region as the boundary ∂A, and denote the internal fringe $\ell^i A$ of a simple fuzzy region by the internal boundary $\partial^i A$; and denote the frontier by $\partial^e A$.

Proposition 4.23 *Let A be a simple fuzzy region in a connected fts X.*
(1) *The interior of the boundary* $(\partial A)^o$ *is regular open*;
(2) *The core of the boundary of A is empty:* $(\partial A)^{\oplus} = \varnothing$;
(3) *The boundary of the boundary is equal to the boundary of the boundary of A:* $\partial A = \partial(\partial A)$;
(4) *The boundary is equal to the union of the boundary of the core and the boundary of the closure of A*: $\partial A = \partial(A^{\oplus}) \cup \partial(A^{-})$.

Proof:
(1) Since $A^{\oplus}$ is regular open, $A^{\oplus c}$ is regular closed. And $A^{\oplus co}$ is regular open. Therefore $(\partial A)^o = (\ell A)^o = (A \cap A^{\oplus c})^o = A^o \cap A^{\oplus co}$. $(\partial A)^o$ is regular open.
(2) $(\partial A)^{\oplus} = (\ell A)^{\oplus} = \varnothing$.
(3) $\partial A = (\partial A)^{\oplus} \cup \partial(\partial A) = \partial(\partial A)$.
(4) Since *A* is closed, $\partial A = \partial(A^{-})$. Since $A^{\oplus} \cap \partial(A^{\oplus}) = \varnothing$, $A^{\oplus} \cap \ell A = \varnothing$, and ℓA is closed, we have $\partial(A^{\oplus}) \subseteq \ell A = \partial A$. Therefore, $\partial A = \partial(A^{\oplus}) \cup \partial(A^{-})$.

Proposition 4.24 *Let A be a simple fuzzy region in a connected fts X. The interior of the boundary of the core of A is empty, i.e.,* $(\partial(A^{\oplus}))^o = \varnothing$.

Proof: $l(A^{\oplus}) = \partial(A^{\oplus})$ according to Proposition 4.9. According to proposition 4.5(5) we have $A^{\oplus\oplus} \cap \ell(A^{\oplus}) = A^{\oplus} \cap \ell(A^{\oplus}) = \varnothing$. If $(\ell(A^{\oplus}))^o \neq \varnothing$, then $A^{\oplus o} \cup (\ell(A^{\oplus}))^o \supseteq A^{\oplus o}$ and $A^{\oplus}$ is not regular open. Contradiction! Therefore, $(\partial(A^{\oplus})) = (\ell(A^{\oplus}))^o = \varnothing$.

4.4.5 A simple fuzzy region in $\widetilde{R}^2$

In Section 2.5.9, we have defined the fuzzy Euclidean space $\widetilde{R}^2$, which is induced from the cts with usual topology. Figure 4.12 shows a simple fuzzy region A in $\widetilde{R}^2$. In Figure 4.12(1), the membership values of the simple fuzzy region A are represented by the height. Figure 4.12(2) is the planar representation so that the topological properties can be visualized more clearly. By such a definition, not only may the membership values at the boundary of the core be equal to 1, but there may also be some points whose membership values are equal to 1 although they do not belong to the core. In Figure 4.12 the membership values on the line and the point are 1 but the line and the point belong to the fringe of the simple fuzzy region. The membership values of points of A's interior on the line and the point are less than 1, and the membership values of points on the closure of A's interior on the line and the point are also less than 1, but the membership values of points on A are equal to 1. Therefore the closure of its interior is less than A; A is not regular closed.

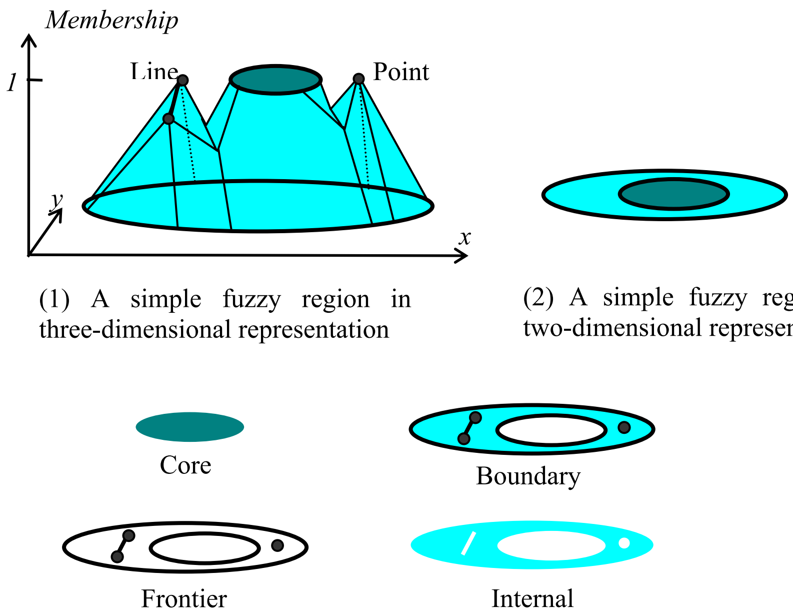


Figure 4.12 A simple fuzzy region in $\widetilde{R}^2$

Figure 4.13 lists some impossible settings for a simple fuzzy region in $\widetilde{R}^2$. In Figure 4.13(1), there is a point whose membership value is 0. It is not a simple fuzzy region since the interior is not regular open. Figure 4.13(2) shows a point whose membership value is less than 1 in the "area" of the core. The interior and the core are not regular open. In Figure 4.13(3) the fuzzy set is not regular closed. Figure 4.13(4) shows that the outer is not open-connected. In Figure 4.13(5) the boundary is not double-connected. Figure 4.13(6) shows that the internal fringe is not double-connected.

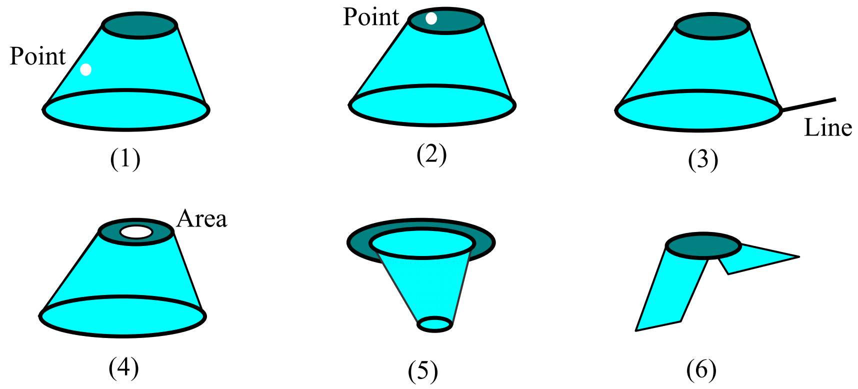


Figure 4.13 Some impossible settings of a simple fuzzy region in $\widetilde{R}^2$

4.5 Topological relations between two simple fuzzy regions in $\widetilde{R}^2$

4.5.1 Forms of intersection matrices in $\widetilde{R}^2$

Since the fringe is equal to the boundary of a simple fuzzy region *i.e.*, $\partial A = \ell A$ according to Proposition 4.9, then the 3*3-intersection matrix (1) can be written as:

$$I_{3*3} = \begin{bmatrix} A^{\oplus} \cap B^{\oplus} & A^{\oplus} \cap \partial B & A^{\oplus} \cap B^{=} \\ \partial A \cap B^{\oplus} & \partial A \cap \partial B & \partial A \cap B^{=} \\ A^{=} \cap B^{\oplus} & A^{=} \cap \partial B & A^{=} \cap B^{=} \end{bmatrix} \quad (3)$$

And the intersection matrix (1') can be written as:

$$I_{3*3} = \begin{bmatrix} A^{i} \cap B^{i} & A^{i} \cap \partial^{e} B & A^{i} \cap B^{=} \\ \partial^{e} A \cap B^{i} & \partial^{e} A \cap \partial B & \partial^{e} A \cap B^{=} \\ A^{=} \cap B^{i} & A^{=} \cap \partial^{e} B & A^{=} \cap B^{=} \end{bmatrix} \quad (3')$$

The 4*4-intersection matrix can be written as:

$$I_{4*4} = \begin{bmatrix} A^{\oplus} \cap B^{\oplus} & A^{\oplus} \cap \partial^{e} B & A^{\oplus} \cap \partial^{i} B & A^{\oplus} \cap B^{=} \\ \partial^{e} A \cap B^{\oplus} & \partial^{e} A \cap \partial^{e} B & \partial^{e} A \cap \partial^{i} B & \partial^{e} A \cap B^{=} \\ \partial^{i} A \cap B^{\oplus} & \partial^{i} A \cap \partial^{e} B & \partial^{i} A \cap \partial^{i} B & \partial^{i} A \cap B^{=} \\ A^{=} \cap B^{\oplus} & A^{=} \cap \partial^{e} B & A^{=} \cap \partial^{i} B & A^{=} \cap B^{=} \end{bmatrix} \quad (4)$$

For GIS applications, Warren's boundary is adopted as the boundary for a simple fuzzy region. Therefore, if the topological relations between two simple fuzzy regions are identified based on the 3*3-intersection matrix, form (3) is more suitable than form (3'). Form (4) can be adopted if the topological relations are identified based on the 4*4-intersection matrix.

4.5.2 Some limitations on topological relations

In general, the 3*3-intersection matrix will deduce 2^9=512 topological relations between two fuzzy sets in the fts. Since a simple fuzzy region is not a general fuzzy set, some limitations exist between two fuzzy regions. The following propositions show these limitations.

Proposition 4.25 *Let A and B be two simple fuzzy regions in a connected fts X. If A's core does not intersect with B's core, then A's core does not intersect with the boundary of B's core, and B's core does not intersect with the boundary of A's core, i.e., if* $A^{\oplus} \cap B^{\oplus} = \varnothing$, *then* $A^{\oplus} \cap \partial(B^{\oplus}) = \varnothing$ *and* $B^{\oplus} \cap \partial(A^{\oplus}) = \varnothing$.

Proof: Suppose *B*'s core intersects with the boundary of $A^{\oplus}$, that is $B^{\oplus} \cap \partial(A^{\oplus}) = D$. Then $D^o = \varnothing$ according to Proposition 4.24. This means, $\forall x \in X$, $D^o(x) = 0$. Then $\forall x_\lambda \in D$, all pan-neighborhoods of x_λ intersect with $A^{\oplus}$ at x_λ, as well as *y* where $\text{supp(y)} \neq \text{supp}(x_\lambda)$. On the other hand, since $D \subseteq B^{\oplus}$, $B^{\oplus}$ is a pan-neighborhood of x_λ. Therefore, $A^{\oplus} \cap B^{\oplus} \neq \varnothing$. Contradiction! The other part of the proposition can be proven in the same way.

Proposition 4.26 *Let A and B be two simple fuzzy regions in a connected fts X. If A's core intersects with the boundary of B's core, then it must intersect with B's core, i.e., if* $A^{\oplus} \cap \partial(B^{\oplus}) \neq \varnothing$, *then* $A^{\oplus} \cap B^{\oplus} \neq \varnothing$.

Proof: Suppose *A*'s boundary cannot intersect with *B*'s boundary, then *A*'s boundary cannot intersect with the boundary of *B*'s core according to Proposition 4.26. Contradiction!

Proposition 4.27 *Let A and B be two simple fuzzy regions in a connected fts X. If A's core intersects with B's core and outer, then it must also intersect with B's boundary, i.e., if* $A^{\oplus} \cap B^{\oplus} \neq \varnothing$, *and* $A^{\oplus} \cap B^{=} \neq \varnothing$, *then* $A^{\oplus} \cap \partial B \neq \varnothing$.

Proof: We will show that *B*'s core is disjoint with *B*'s outer. Suppose $A^{\oplus} \cap B^{\oplus} = C \neq \varnothing$ and $A^{\oplus} \cap B^{=} = D \neq \varnothing$, but $A^{\oplus} \cap \partial(B^{\oplus}) = \varnothing$. Then $C \cup D = A^{\oplus}$ since *C* is crisp and $B^{=}$ is crisp. Since *C* and *D* are open, therefore $A^{\oplus}$ is not open-connected. Contradiction!

Proposition 4.28 *Let A and B be two simple fuzzy regions in a connected fts X. If A's core is disjoint with B's core, and A's boundary intersects with B's core, then A's boundary intersects with B's boundary. That is if* $A^{\oplus} \cap B^{\oplus} = \varnothing$ *and* $\partial A \cap B^{\oplus} \neq \varnothing$, *then* $\partial A \cap \partial B \neq \varnothing$.

Proof: Suppose $A^{\oplus} \cap B^{\oplus} = \varnothing$ and $\partial A \cap B^{\oplus} \neq \varnothing$ but $\partial A \cap \partial B = \varnothing$. Then $B^{\oplus} \cap \partial(A^{\oplus}) = \varnothing$ and $A \cap \partial(B^{\oplus}) = \varnothing$ according to Proposition 4.24. Then $A \cap B^{\oplus} = \varnothing$. Therefore, $\partial A \cap B^{\oplus} = \varnothing$. Contradiction!

Proposition 4.29 *Let A and B be two simple fuzzy regions in a connected fts X. If A's core and boundary intersect with B's boundary, then the boundary of A's core intersects with B's boundary. That is, if* $A^{\oplus} \cap \partial B \neq \varnothing$ *and* $\partial A \cap \partial B \neq \varnothing$, *then* $\partial B \cap \partial(A^{\oplus}) \neq \varnothing$.

Proof: Let $A^{\oplus} \cap \partial B \neq \varnothing$ and $\partial A \cap \partial B \neq \varnothing$. Suppose $\partial B \cap \partial(A^{\oplus}) = \varnothing$. Then $\partial B \cap A^{\oplus -} = C$ is closed and $\partial B \cap A^{\oplus c} = D$ is closed. Then $\partial B = C \cup D$. Contradiction!

4.5.3 Topological relations based on empty/non-empty contents in $\widetilde{R}^2$

Topological relations between two simple fuzzy regions can be identified by using the 3*3-intersection matrix (3) in the fts. We will identify the topological relations between two fuzzy regions in fuzzy Euclidean space $\widetilde{R}^2$. The fuzzy region is further limited to a two-dimensional bounded set. The following conditions, which hold between the topological properties of two crisp regions in the crisp fts, also hold in $\widetilde{R}^2$ according to the above propositions:

(1) The outers of two fuzzy regions intersect each other;
(2) Any part of one fuzzy region must intersect at least one part of the other fuzzy region, and vice versa;
(3) If one fuzzy region's core intersects the other's core and outer, then it must also intersect the other's boundary, and vice versa;
(4) If both cores are disjoint, then one fuzzy region's core intersects the other's boundary, or the other's outer, and vice versa;
(5) If both cores are disjoint and one fuzzy region's boundary intersects the other's core, then the two boundaries must intersect each other, and vice versa;
(6) If one fuzzy region's core intersects the other's outer, then its boundary must also intersect the other's outer, and vice versa;
(7) If one fuzzy region's core is a subset of the other fuzzy region, then its boundary must intersect the other, and vice versa;
(8) If one fuzzy region's core does not intersect the other fuzzy region, then its core must intersect the other's outer, and vice versa;
(9) If both cores do not intersect each other, then at least one boundary must intersect its opposite outer;
(10) If both boundaries intersect the opposite cores, then the boundaries must also intersect each other;
(11) If one fuzzy region's boundary intersects the other's core and outer, then it must also intersect the other's boundary, and vice versa;
(12) If one fuzzy region is a subset of the core of the other, then its outer must intersect the other's core, and vice versa.

Under such conditions, 44 relations between two regions in $\widetilde{R}^2$ can be identified (refer to Appendix 1). The number is the same as for the approach in Chapter 3 and the algebraic model by Clementini and De Felice (1996). If two simple fuzzy regions degenerate into simple crisp regions, then only eight relations can be identified. If the 4*4-intersection matrix is adopted, then 152 topological relations can be identified (refer to Appendix 2) in $\widetilde{R}^2$ by changing the interior, the boundary of the boundary, the interior of the boundary and the exterior into the core, the frontier, the internal boundary and the outer, under the 12 conditions that are listed in Section 3.7.2.

4.5.4 More topological invariants

The above interpretation of topological relations is based on the topologically invariant "empty/non-empty" dichotomy. When the intersection is empty, *i.e.*, $A \cap B = \varnothing$, it means $\forall x \in X$, if $A(x) > 0$, then $B(x) = 0$. This result is the same as in a cts. In cts, when the intersection is non-empty, *i.e.*, $A \cap B = C \neq \varnothing$, we have $\forall x \in C$, $A(x) = 1$, and $B(x) = 1$. For example, if $\partial A \cap \partial B = C \neq \varnothing$ in the cts, then whenever $C(x) = 1$ we have $\partial A(x) = \partial B(x) = 1$.

In the cts, the comparison can only be done crisply. However, there are more requirements when spatial objects are represented fuzzily. For example, there are two fuzzy spatial objects: grassland and bush. Someone would like to know which one is the dominating feature on a piece of land. Therefore, the comparison should be done at the level of membership values of these two objects.

Based on the above definitions and the intersection matrices, it is possible to identify more topological relations to answer these kinds of questions.

Definition 4.30 *Let A and B be two simple fuzzy regions in a connected fts X, and* $A \cap B = C \neq \varnothing$. *Denote* $A|_c = A \cap \mathrm{supp}(C)$ *and* $B|_c = B \cap \mathrm{supp}(C)$. *Define four comparative (relationships)* $\neq, \supseteq, \subseteq, =$ *between* $A|_c$ *and* $B|_c$*:*

(1) $A|_c \neq B|_c$: $\exists x \in X$, $A|_c(x) > B|_c(x)$ *and* $\exists y \in X$, $A|_c(y) < B|_c(y)$;

(2) $A|_c \supseteq B|_c$: $\forall x \in X$, $A|_c(x) \geq B|_c(x)$ *and* $\exists y \in X$, $A|_c(y) > B|_c(y)$

(3) $A|_c \subseteq B|_c$: $\forall x \in X$, $A|_c(x) \leq B|_c(x)$ *and* $\exists y \in X$, $A|_c(y) < B|_c(y)$

(4) $A|_c = B|_c$: $\forall x \in X$, $A|_c(x) = B|_c(x)$

Figure 4.14 shows the intersection between *A* and *B*.

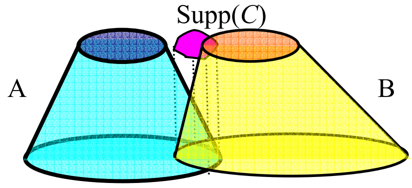


Figure 4.14 Intersection between two simple fuzzy regions

These four comparatives are topological invariants. Let f be a (fuzzy) homeomorphic mapping. Since f is point preserving, *i.e.*, $f(x_\lambda) = (f(x))_\lambda$, if $A|_c(x) \geq B|_c(x) > 0$, then $f(A|_c(x)) \geq f(B|_c(x)) > 0$. The remaining comparatives can be proved in the same way.

For two simple fuzzy regions A and B, write $A^\oplus|_c$, $\partial A|_c$, and $A^=|_c$ for the intersection parts of $A^\oplus$, ∂A and $A^=$, and $B^\oplus|_c$, $\partial B|_c$, $B^=|_c$ for the intersection parts of $B^\oplus$, ∂B, and $B^=$, respectively. When the nine intersections $A^\oplus \cap B^\oplus$, $A^\oplus \cap \partial B$, $A^\oplus \cap B^=$, $\partial A \cap B^\oplus$, $\partial A \cap \partial B$, $\partial A \cap B^=$, $A^= \cap B^\oplus$, $A^= \cap \partial B$ and $A^= \cap B^=$ are non-empty, the comparison between $A|_c$ and $B|_c$ offers different options for mutual relations.

(1) There is only one relation between $A^\oplus|_c$ and $B^\oplus|_c$: $A^\oplus|_c = B^\oplus|_c$, since $\forall x \in X\ A^\oplus|_c(x) = B^\oplus|_c(x) = 1$.

(2) Four relations exist between $\partial A|_c$ and $\partial B|_c$: $\partial A|_c \supseteq \partial B|_c$, $\partial A|_c \subseteq \partial B|_c$, $\partial A|_c = \partial B|_c$ and $\partial A|_c \neq \partial B|_c$ when the boundaries are not crisp.

(3) There is only one relation between $A^\oplus|_c$ and $\partial B|_c$: $A^\oplus|_c \supseteq \partial B|_c$ since $A^\oplus|_c(x) = 1$ when $A^\oplus|_c(x) > 0$, and $\partial B|_c(x) = 1$ cannot hold for $\forall x_\lambda \in \partial B|_c$.

(4) Similar to (3), there is only one relation between $\partial A|_c$ and $B^\oplus|_c$: $\partial A|_c \subseteq B^\oplus|_c$.

(5) There is only one relation between $\partial A|_c$ and $B^=|_c$: $\partial A|_c \subseteq B^=|_c$. Correspondingly, $A^=|_c \supseteq \partial B|_c$ between $A^=|_c$ and $\partial B|_c$.

(6) There is only one relation between $A^\oplus|_c$ and $B^=|_c$: $A^\oplus|_c = B^=|_c$. Correspondingly, $A^=|_c = B^\oplus|_c$ between $A^=|_c$ and $B^\oplus|_c$.

(7) There is only one relation between $A^=|_c$ and $B^=|_c$: $A^=|_c = B^=|_c$.

Based on these options, the 3*3-intersection matrix for two simple fuzzy regions A and B can be adjusted. Denote the empty intersection by 0. The possibilities for topological relations are described in the following matrix:

$$I_{3*3} = \begin{bmatrix} 0/= & 0/\supseteq & 0/= \\ 0/\subseteq & 0/\supseteq/\subseteq/\neq/= & 0/\subseteq \\ 0/= & 0/\supseteq & = \end{bmatrix} \tag{5}$$

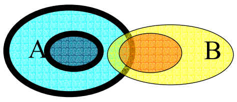
There are 41 non-empty and three empty sets (Relation no. 1, 39 and 40) between the boundary-boundary intersections in the 44 relations between two simple fuzzy regions (see Appendix 1). Therefore, between two simple fuzzy regions there are 41*4 + 3 = 167 possible topological relations. When the simple fuzzy regions degenerate into simple crisp regions, the relations between boundaries will be "=".

4.5.5 Topological relations between two real simple fuzzy regions in $\widetilde{R}^2$

The above 167 topological relations are realizable between two simple fuzzy regions. However, some of them correspond to extreme conditions. For example, the sixth relation can be decomposed into four relations (Figure 4.15).

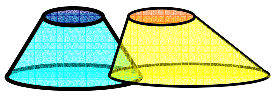
In Figure 4.15, the intersection between $\partial A \cap \partial B$ includes four settings: $\partial A \neq \partial B$ (6a), $\partial A \subseteq \partial B$ (6b), $\partial A \supseteq \partial B$ (6c) and $\partial A = \partial B$ (6d). Relations (6c) and (6d) are realizable under extreme conditions. Relation (6d) requires that $\forall x_\lambda \in \partial A \cap \partial(B^{\oplus})$, $\partial A(x) = 1$. This occurs very seldom in GIS applications since it asks that A's boundary has some crisp points. In order to avoid this case, we can refine the definition of the simple fuzzy region into a real simple fuzzy region.

$$(6)\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

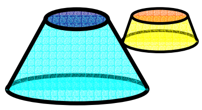


Relation (6) in Appendix 3.1

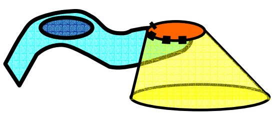
$$(6a)\begin{bmatrix} 0 & 0 & = \\ \subseteq & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$$



$$(6b)\begin{bmatrix} 0 & 0 & = \\ \subseteq & \subseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$$



$$(6c)\begin{bmatrix} 0 & 0 & = \\ \subseteq & \supseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$$



$$(6d)\begin{bmatrix} 0 & 0 & = \\ \subseteq & = & \subseteq \\ = & \supseteq & = \end{bmatrix}$$

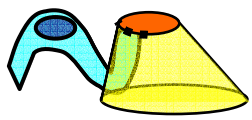


Figure 4.15 Four settings of Relation (6) in $\widetilde{R}^2$

Definition 4.31 *Let A be a simple fuzzy region in a connected fts. A is called a real simple fuzzy region if (1)* $\forall x \in X$, $0 < (A \cap A^{\oplus -c})(x) < 1$, and $\partial(A^{\oplus})(x) = 1$*; (2)* $A^{=-} \cap A^{\oplus -} = \varnothing$.

A real simple fuzzy region cannot degenerate into a simple crisp region, since $A \cap A^{\oplus -}$ is not empty. Condition (1) indicates that the boundary of a fuzzy region contains some points that are not crisp. Condition (2) shows that the complement of the interior of the support of a fuzzy region (which can also be regarded as a simple crisp region) cannot intersect the closure of the core of the region (which is also a simple crisp region). In other words, X is Q-separated into the core and the outer by the support of the boundary. Figure 4.16 shows a real simple fuzzy region in $\widetilde{R}^2$.

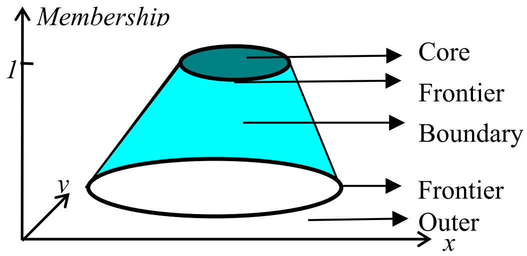


Figure 4.16 A real simple fuzzy region in $\widetilde{R}^2$

Proposition 4.32 *Let A and B be two real simple fuzzy regions. If A's boundary intersects with B's core and boundary, respectively, then* $\partial A|_c = \partial B|_c$ *and* $\partial A|_c \supseteq \partial B|_c$ *between the boundary of A and the boundary of B are not realizable.*

Proof: Let $\partial A \cap \partial B = C \neq \varnothing$ and $\partial A \cap B^{\oplus} = D \neq \varnothing$. Then $\partial A \cap \partial(B^{\oplus}) \neq \varnothing$ according to Proposition 4.29. Let $\partial A \cap \partial(B^{\oplus}) = E$. Assume that $\partial A|_c = \partial B|_c$, then $\partial A|_E(x) = 1$ since $\partial(B^{\oplus})|_E(x) = 1$ $\forall x \in X$. Then $\partial A|_E(x) = \partial(A^{\oplus})(x)$ according to condition (1) of Definition 4.32. (1) When $B^{\oplus} \cap A^{+} \neq B^{\oplus}$, then $B^{\oplus} \cap A^{=} \neq \varnothing$ and $B^{\oplus} \cap \partial(A^{=}) \neq \varnothing$. Since $B^{\oplus} \cap \partial A \neq \varnothing$ it follows that $\partial(B^{\oplus}) \cap A^{=} \neq \varnothing$ and $\partial(B^{\oplus}) \cap \partial(A^{=}) \neq \varnothing$. That is, $\exists y \in X$, $\partial(B^{\oplus})(y) \wedge \partial(A^{=})(y) \neq 0$ and $\partial(A^{=})$ intersects ∂A on y. Then $\partial(A^{\oplus}) \cap A^{=-} \neq \varnothing$. Contradiction! (2) When $B^{\oplus} \cap A^{+} = B^{\oplus}$, then $\forall x_{\lambda} \in \partial(B^{\oplus})|_E$. We have $\partial(A^{\oplus})(x) = \partial(B^{\oplus})(x)$. Suppose there is a subset P of $\partial(B^{\oplus})$ such that $P \cap \partial A = \varnothing$. Then $P \subseteq A^{\oplus}$, $A^{\oplus} \supseteq B^{\oplus}$, and $\partial A \cap B^{\oplus} = \varnothing$. Contradiction! Therefore, the assumption $\partial A|_c = \partial B|_c$ is wrong and $\partial A|_c = \partial B|_c$ between the boundary of A and the boundary of B is not realizable. If we change the assumption $\partial A|_c = \partial B|_c$ to $\partial A|_c \supseteq \partial B|_c$, the above procedure shows that it cannot hold either.

Similarly, if B's boundary intersects A's core and boundary, respectively, then $\partial A|_c = \partial B|_c$ and $\partial A|_c \subseteq \partial B|_c$ between the boundary of A and the boundary of B are not realizable.

Under the limitation of Proposition 4.32, 77 topological relations between two real simple fuzzy regions can be derived (see Appendix 3). The topological invariants of relationships expand the topological relations between two regions from the "horizontal" level to the "vertical" level, where *horizontal* refers to whether the intersection of two topological parts is empty or not, and *vertical* refers to the comparison between membership values of two topological parts when there is a non-empty intersection.

4.6 Comparisons

4.6.1 Comparison of topological spaces

We have introduced several spaces: crisp topological space (R^2, d), crisp fuzzy topological space $(\widetilde{R}^2, C)$, and induced fuzzy topology space $(\widetilde{R}^2, \delta)$ or $\widetilde{R}^2$. The power set of $\widetilde{R}^2$ of cts $(\widetilde{R}^2, C)$ is equal to the power set of $\widetilde{R}^2$ of induced fts $(\widetilde{R}^2, \delta)$. It contains the power set of R^2 of cts (R^2, d), since the power set of $\widetilde{R}^2$ contains fuzzy sets but the power set of R^2 is just the collection of crisp subsets of R^2. However, the topology C of $(\widetilde{R}^2, C)$ is equal to the topology d of (R^2, d). They are coarser than the induced topology δ of $(\widetilde{R}^2, \delta)$. The topology C of $(\widetilde{R}^2, C)$ is actually a special case of $(\widetilde{R}^2, \delta)$. They are illustrated in Figure 4.17.

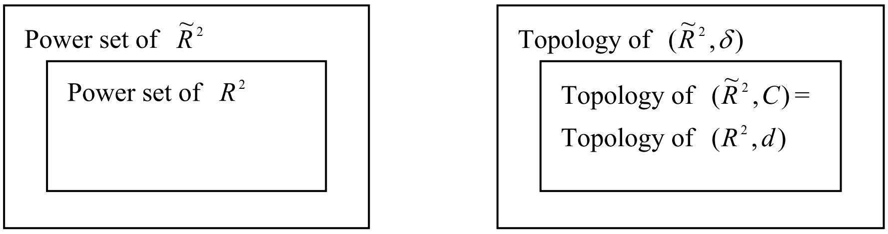


Figure 4.17 Comparisons between power sets and topologies

4.6.2 Comparison of definitions

We have also introduced two definitions of a simple fuzzy region. Definition (1) in Chapter 3 is based on the crisp fts, and definition (2) in this chapter is based on a general fts. Although definition (2) is defined in the general fts, it should be noticed that some fuzzy sets may be a simple fuzzy region in $(\widetilde{R}^2, C)$, but not a simple fuzzy region in $(\widetilde{R}^2, \delta)$. For example, define a fuzzy set A to be a fuzzy closed disk in $\widetilde{R}^2$:

$$A(x) = \begin{cases} 1 & if \quad r < 1 \\ 0.5 & if \quad 1 \le r \le 2 \\ 0 & if \quad r > 2 \end{cases}$$

Then A is not closed in the induced fts $(\widetilde{R}^2, \delta)$ (Figure 4.18). However, the support of A in crisp fts $(\widetilde{R}^2, C)$ is closed. Therefore, it is a simple fuzzy region in $(\widetilde{R}^2, C)$, but it is not a simple fuzzy region in the induced fts $(\widetilde{R}^2, \delta)$ since A is not an upper semicontinuous mapping from R^2 to [0,1].

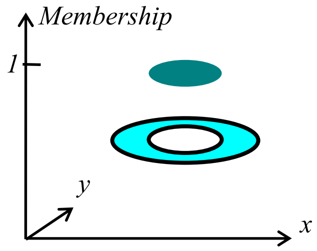


Figure 4.18 A fuzzy disk is a simple fuzzy region in $(\widetilde{R}^2, C)$

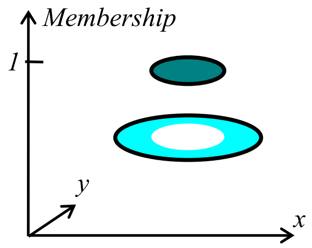


Figure 4.19 A fuzzy disk is a simple fuzzy region in $(\widetilde{R}^2, \delta)$

In $(\widetilde{R}^2, \delta)$, A will be a simple fuzzy region if

$$A(x) = \begin{cases} 1 & if \quad r \le 1 \\ 0.5 & if \quad 1 < r \le 2 \\ 0 & if \quad r > 2 \end{cases}$$

This is more suitable for the real GIS applications (Figure 4.19).

Definition (1) is similar to the definition of a region with a broad boundary that was defined by Clementini and Di Felice (1996). Since their definition is defined in the cts, the broad boundary can be regarded as a crisp closed set in the cts, which is projected from the fuzzy set onto the crisp plane.

Definition (1) is just based on the crisp fts, in which the membership values of fuzzy sets are neglected since every open set and every closed set are crisp. Definition (2) is defined in a real fts, such that it considers the fuzzy sets to be open or closed. Therefore definition (2) will be adopted, and definition (1) can be regarded as a draft version of a simple fuzzy region.

4.6.3 Comparison of different approaches

We have also established several intersection matrices in the fts to identify the topological relations between fuzzy sets. The intersection matrices (1) and (1') will be equal to intersection matrix (1) in Chapter 3 when an fts is a crisp fts. Therefore, the intersection matrix (1) in Chapter 3 can be regarded as a special form of the intersection matrices (1) and (1') in the general fts.

Although definition (1) is a draft version of the formal definition of a simple fuzzy region, the topological relations between two simple fuzzy regions based on definition (1) are the same as the topological relations between two simple fuzzy regions based on definition (2) when the empty/non-empty topological invariants are adopted. This is because, if a fuzzy set A is a simple fuzzy region in $(\widetilde{R}^2,\delta)$, the support of A is still a simple fuzzy region in $(\widetilde{R}^2,\delta)$. Therefore, the topological matrices are universally applicable no matter in which fts a simple fuzzy region is defined.

The shortcoming of the 3*3-intersection matrix and the 4*4-intersection matrix defined in Chapter 3 is that they cannot be adopted for identifying the topological relations between two simple fuzzy regions in terms of membership values. This is the same in the algebraic model. Intersection matrix (1) can identify more topological relations between two simple fuzzy regions than either the intersection matrix (1) in Chapter 3 or the 9-intersection matrix.

The above comparisons can be summarized as follows:

(1) Simple fuzzy regions can be defined in different topological spaces;
(2) The intersection matrices can be derived from different topological spaces. All the 3*3-intersection matrices and the 4*4-intersection matrix are applicable to identify the topological relations between two simple fuzzy regions no matter in which topological space a simple fuzzy region is defined.
(3) The intersection matrices in this chapter can identify more topological relations than the intersection matrices in Chapter 3.

4.7 Conclusions and discussions

This chapter proposes another approach for identifying topological relations between simple fuzzy regions. The basic idea is to derive the topological properties that are mutually disjoint in a general fts. Many topological properties are found and they are mutually disjoint. Two 3*3-intersection matrices and a 4*4-intersection matrix are introduced in a general fts. These matrices can be adopted to derive the topological relations between two fuzzy sets.

Another contribution is the formal definition of a simple fuzzy region in a general fts. The definition is based on the core, the internal fringe, the frontier and the outer of a fuzzy set in the fts. This definition is better applicable for GIS applications than the definition in Chapter 3 that is derived based on the interior and the boundary and the exterior of a fuzzy set in the crisp fts, because it considers a finer structure of a fuzzy set in the fts. The definition of a simple fuzzy region in Chapter 3 can be regarded as a draft version of the definition of a simple fuzzy region.

Three intersection matrices are derived based on the topological properties of two fuzzy sets. Intersection matrix (1) is more suitable than intersection matrix (1') for identifying the topological relations between two simple fuzzy regions. The 4*4-intersection matrix can be formulated if the fringe is decomposed into the internal fringe and the frontier of a fuzzy set.

In general, these intersection matrices are different from the intersection matrices derived in Chapter 3, since they are based on different topological properties. However, they are all applicable for deriving topological relations between two simple fuzzy regions no matter in which topological space a simple fuzzy region is defined.

Forty-four (44) and 152 topological relations can be identified between two simple fuzzy regions in the induced fuzzy topological space of the usual Euclidean space.

In practice, there are more requirements when spatial objects are represented fuzzily. Since the intersection matrices are derived in the general fts, it is possible to identify more topological relations according to the membership values of fuzzy spatial objects. Based on the topological invariants of four comparisons in the fts, 77 relations are identified between two real simple fuzzy regions.

The chapter investigates only topological relations between two simple fuzzy regions in $\widetilde{R}^2$. They have to be extended to more complex fuzzy regions. Further research into relations between spatial objects of different dimensions is necessary to model all kinds of fuzzy spatial objects, including points, lines and regions, which will be shown in Chapter 5.

Chapter Five

Modeling Fuzzy Spatial Objects and their Topological Relations

5.1 Introduction

Chapters 3 and 4 discussed the definition of a simple fuzzy region in a general fts and the topological relations between two simple fuzzy regions in the fuzzy Euclidean space $(\widetilde{R}^2, \delta)$, whose topology is induced from the crisp Euclidean space R^2. However, the topological relations between fuzzy point, fuzzy line and fuzzy region have not been formalized as yet. In conventional GIS, a common approach to modeling spatial objects is oriented towards algebraic topology. In general, algebraic topology is a branch of topology that studies topology by using the algebraic method. In algebraic topology, the most primitive concepts are simplex and simplicial complex, as well as the cell and cell complex. These concepts have been used to model (crisp) spatial objects (Molenaar 1998).

In order to model fuzzy spatial objects in GIS, these concepts have to be extended from the crisp domain to the fuzzy domain. This chapter will introduce a data model to represent fuzzy spatial objects, including fuzzy points, fuzzy lines, and fuzzy regions by using algebraic topology. Many properties related to fuzzy spatial objects have been discussed, such as area, perimeter, length and distance (Rosenfeld 1985a, 1985b, Schneider 2000). This chapter will focus on the topological relations between spatial objects (Tang *et al.* 2003b). Section 5.2 reviews some basic concepts of algebraic topology and the conventional spatial modeling techniques based on algebraic topology. Section 5.3 proposes the definitions of fuzzy cell and fuzzy cell complex. Section 5.4 provides a data model to represent fuzzy spatial objects, including fuzzy points, fuzzy lines and fuzzy regions. Section 5.5 identifies the topological relations between these fuzzy spatial objects. Section 5.6 presents conclusions and discussions.

5.2 Algebraic topology and spatial data modeling

5.2.1 Simplex and simplicial complex

The main idea of algebraic topology is to consider two topological spaces to be equivalent if they have "the same shape" in a sense that is much broader than homeomorphisms (Hatcher 2002). In general, algebraic topology can be separated into the two branches of homology and homotopy theory. With regard to spatial data modeling in this thesis, the whole theory is not needed; instead, we just discuss some basic concepts of algebraic topology.

In algebraic topology, some complicated spaces are built based on "bricks". One of these bricks is a simplex. Intuitively speaking, a simplex is the simplest geometric figure of a respective geometric dimension, *i.e.*, a point in a zero-dimensional space, a straight line segment in a one-dimensional space, a triangle in a two-dimensional space (Kainz 2004).

Formally, a *p*-simplex s_p in R^n is generated by a point set $(x_0, x_1, ..., x_p)$, where $p \le n$:

$$s_p = \sum_{i=0}^{p} \lambda_i x_i, \, where \sum_{i=0}^{p} \lambda_i = 1, \forall i \; \lambda_i \ge 0$$

We also write a *p*-simplex s_p as a *p-dimensional simplex*. The points x_i are the *vertices of simplex* s_p. The numbers λ_i are called the *barycentric coordinates* of a simplex. The points where all barycentric coordinates are greater than 0 are called the *internal points* of the simplex. The remaining points, where at least one of the barycentric coordinates is 0, are called *the edge points* of the simplex. The set of all edge points is called *the edge* of the simplex. Let *s*, *t* be two simplexes and $k \le p$. If all vertices of simplex t_k are vertices of simplex s_p, then t_k is called *a k-face of* s_p. If $k < p$, then t_k is called *a proper face of* s_p.

Figure 5.1 shows a zero-dimensional simplex, one-dimensional simplex, and two-dimensional simplex.

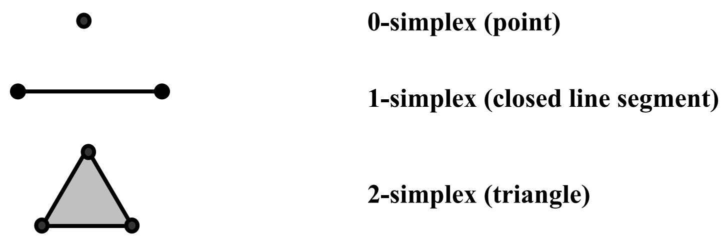


Figure 5.1 0-simplex, 1-simplex and 2-simplex

In Figure 5.2, the two 0-simplexes are the proper faces of the 1-simplex; and a 2-simplex has three zero-dimensional proper faces and three proper one-dimensional faces.

Let S be a finite set of simplexes that fulfils the following conditions:

(1) If the simplex s_p is an element of S, then each face of s_p belongs S;

(2) For any two simplexes in S, the intersection of these two simplexes is either empty or a common face.

Then S is called a *simplicial complex*. A finite simplicial complex is a simplicial complex if the collection of simplexes is finite.

A simplicial complex is not a topological space. However, simplexes are subsets of R^n. Let's investigate the topology of a finite complex K. Let $|K|$ be the subset that is the union of finite simplexes of K in R^n. Giving each simplex s its natural topology as a subspace of R^n (where n is the largest dimension of all simplexes), we then topologize $|K|$ by declaring a subset A of $|K|$ to be closed in $|K|$ if $A \cap s$ is closed in s for each s in K. It is easy to see that this defines a topology on $|K|$, because any finite unions and arbitrary intersections are still in $|K|$ since A and s are all closed in $|K|$. Since simplexes are finite, the topology of $|K|$ is equivalent to the topology $|K|$ inherited as a subspace of R^n. For suppose $|K|$ is finite and A is closed in $|K|$, then $A \cap s$ is closed in s and hence closed in R^n. Then we can imbed $|K|$ into R^n if K is a finite simplicial complex. The space $|K|$ is called the *polytope* of K.

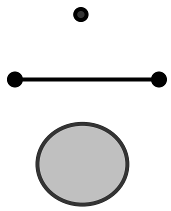
The structure of the simplicial complex can be used to model spatial objects. A more convenient structure is the cell complex, which is built based on cells.

5.2.2 Cell and cell complex

In Chapter 2, an open disk of point x is defined as a set whose radius $r < \varepsilon$ $(\varepsilon \in R^+)$ around x in the Euclidean space R^2. A closed disk of point x is defined as a set whose radius $r \le \varepsilon$ $(\varepsilon \in R^+)$ around x in the Euclidean space R^2. An *n-dimensional open disk* $(D^o)^n$ of point x is a set whose radius $r < \varepsilon$ $(\varepsilon \in R^+)$ around x in the Euclidean space R^n. An *n-dimensional closed disk* D^n of point x is a set whose radius $r \le \varepsilon$ $(\varepsilon \in R^+)$ around x in the Euclidean space R^n. Since point x is arbitrarily chosen, we can neglect point x in the definition of open (or closed) disk. In order to remain consistent with the definition of a simplex, we refer to Munkres's definition of a cell (Munkres 1984, compare with Hatcher 2002, Rotman 1988). An *n-dimensional cell* $\bar{e}^n$ (or *n-cell*) is a space that is homeomorphic to an n-dimensional closed disk D^n. An *n-dimensional open cell* e^n is a space that is homeomorphic to an n-dimensional open disk $(D^o)^n$.

The *closure* of an n-cell is just the n-cell. The *boundary* of an n-cell is defined as the boundary of the n-cell in R^n. The *interior* of an n-cell is the interior of the n-cell in R^n. The interior of an n-cell is equivalent to an open n-cell.

According to the definition, a 0-cell is a point. It is a closed set in R^n. An open 0-cell is still the point. It is open in R^0, but it is not an open set in R^n when $n > 0$. The boundary of a 0-cell is empty. A 1-cell is a closed line segment in R. Its boundary is its two end points. A 2-cell is a closed disk in R^2. Its boundary is a 1-sphere. Figure 5.2 shows these cells.



0-cell (point)

1-cell (closed line segment)

2-cell (closed disk)

Figure 5.2 0-cell, 1-cell and 2-cell

A cell complex X is a space and a collection of disjoint open cells e_i^n whose union is X such that:

(1) X is Hausdorff;

(2) For each open n-cell e_i^n of the collection, there exists a continuous mapping $f_i : D^n \to X$ that maps $(D^o)^n$ homeomorphically onto e_i^n and carries $\partial(D^n)$ into a finite union of open cells, each of dimension less than n;

(3) A set A is closed in X if $A \cap \bar{e}_i^n$ is closed in $\bar{e}_i^n$ for each i.

Let X be a cell complex, and Y be a subspace of X that equals a union of open cells of X. Suppose that for each open cell e_i^n of X contained in Y, its closure is also contained in Y. Y is a closed set in X and it is a cell complex in its own right. It is called a *sub-complex* of X. In particular, the subspace X^p of X that is the union of the open cells of X of dimension at most p satisfies these conditions. It is thus a sub-complex of X, which is called the *p-skeleton of X*.

A *finite cell complex X* is a cell complex for which the collection of open cells is finite. Call each 0,1,…,n-cell in X a *face* of X. A *proper face* is any 0,1,…,(n-1)-cell in X. A finite cell complex can also be defined by the faces of cells. A finite cell complex X is a collection of cells such that (1) every face of a cell of X is in X, and (2) the intersection of two cells is either empty or a common face of both cells.

5.2.3 Spatial data modeling

The structure of cell complex can be easily applied to modeling (crisp) spatial objects. For two-dimensional spatial data modeling, we just consider a complex X is embedded in R^2.

Let *A* be a two-dimensional sub-complex of two-dimensional cells of finite complex *X*. The union of its cells is still two-dimensional. Since the topology of *A* is equivalent to the topology of *A* inherited as the subspace of R^2, we can adopt the interior, the boundary, and the exterior of the union directly in R^2. *A* is called a *simple region in* R^2 such that the union of cells is a connected regular closed set, with a connected boundary and a non-empty connected interior in R^2.

Let *A* be a one-dimensional sub-complex of one-dimensional cells of finite complex *X*. The union of its cells is one-dimensional. The interior, the boundary and the exterior of a sub-complex in *R* is different from the interior, the boundary and the exterior of a sub-complex in R^2. However, the interior, the boundary and the exterior of a sub-complex in *R* are still topological properties of a sub-complex of one-dimensional cells in R^2. We can adopt the interior, the boundary and the exterior of the union in *R*. *A* is called a simple line in R^2 such that the union of cells is connected, it is not self-intersecting and does not form a loop in R^2, and the boundary of the union has two distinct points in *R*.

A is a point if *A* is a 0-cell in R^2.

A simple region, a simple line and a point are shown in Figure 5.3.

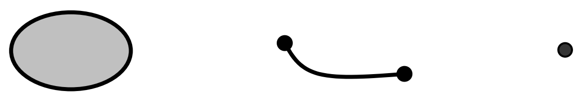


Figure 5.3 Simple region, simple line and point

The topological relations between these spatial objects can be identified based on the 9-intersection matrix. The topological relations between two simple regions can be identified by using the 9-intersection matrix, since the definition of the exterior, the boundary and the interior of a simple region is the same as the exterior, the boundary and the interior of a crisp set in the crisp topological space. The 9-intersection matrix can also identify the topological relations between a simple region and a point. In order to identify the topological relations between a simple region and a simple line, we have to adopt the interior, the boundary, and the exterior of a simple line in *R* for the 9-intersection matrix. The interior, the boundary and the exterior of a simple line in *R* are still topological properties in R^2. This is because the boundary of a simple line in *R* is a set of two points, which is a closed set in R^2, and the interior of a simple line in *R* is the intersection of the simple line with R^2 minus the set of two points. They are still topological properties in R^2.

5.3 Fuzzy cell and fuzzy cell complex

5.3.1 Fuzzy cell

Similar to the definition of an (crisp) *n*-cell, we now extend the notion of cell and cell complex into the fuzzy domain. Define a Euclidean distance between two fuzzy points $p_a(x_1, x_2, ...x_n)$ and $q_b(y_1, y_2, ..., y_n)$ in $\widetilde{R}^n$ by: $d(p_a, q_b) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. Define a fuzzy closed disk D_p^n of a fuzzy point p_a in $\widetilde{R}^n$ by: $D_p^n = \{q_b : d(p_a, q_b) \leq r\}$. Define a fuzzy open disk $(D^n)_p^o$ of a fuzzy point p_a in $\widetilde{R}^n$ by: $(D^n)_p^o = \{q_b : d(p_a, q_b) < r\}$. Let $\widetilde{R}^n$ be equipped with fuzzy topology δ that is induced from the Euclidean topology of R^n. Then a fuzzy open disk $(D^n)_p^o$ is open if it is a lower semicontinuous mapping from R^n to [0,1]. A fuzzy closed disk D_p^n is closed if it is an upper semicontinuous mapping from R^n to [0,1]. $\widetilde{R}^n$ is normal, p-normal and connected.

We define a *fuzzy n-cell* $\widetilde{\widetilde{e}}_n$ by a fuzzy subset of $\widetilde{R}^n$, such that its space is homeomorphic to a fuzzy closed disk and it is an upper semicontinuous mapping from R^n to [0,1]. A fuzzy open *n*-cell $\widetilde{e}_n$ is a fuzzy subset of $\widetilde{R}^n$, such that its space is homeomorphic to a fuzzy open disk and it is a lower semicontinuous mapping from R^n to [0,1].

By definition, *a fuzzy 0-cell* is a fuzzy point. *A fuzzy 1-cell* is a fuzzy closed line segment such that its support is a crisp 1-cell, and the membership function is an upper semicontinuous mapping from *R* to [0,1]. *A fuzzy 2-cell* is a fuzzy closed disk such that its support is a crisp 2-cell, and the membership function is an upper semicontinuous mapping from R^2 to [0,1]. Figure 5.4 shows these fuzzy cells.

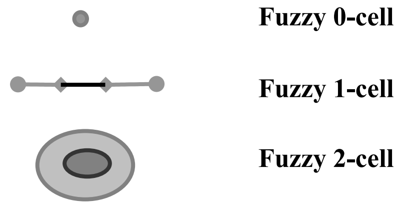


Figure 5.4 Fuzzy 0-cell, 1-cell and 2-cell

A fuzzy *n*-cell is a fuzzy subset of $\widetilde{R}^n$. We adopt the dimension of its support in $\widetilde{R}^n$. That is, a fuzzy cell has dimension *n* if the dimension of its support is *n*.

Unlike the crisp cell structure, a fuzzy *n*-cell has a vertical structure in terms of membership values. We say a fuzzy *n*-cell is a *fuzzy n-sub-cell* of a fuzzy *n*-cell if its support is equal to the support of the fuzzy *n*-cell and the membership value at each of its points is equal to or less than the value on that point of the fuzzy *n*-cell. Figure 5.5

shows a fuzzy 2-cell and its fuzzy 2-sub-cells. A fuzzy *n*-cell may have an infinite number of fuzzy *n*-sub-cells.

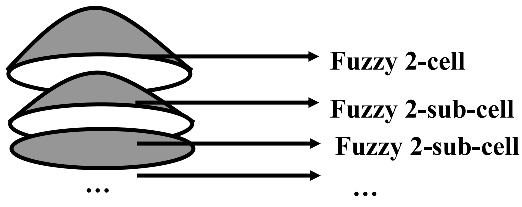


Figure 5.5 Vertical structure of a fuzzy 2-cell

A fuzzy *n*-cell can be regarded as a subset of fuzzy Euclidean space $\widetilde{R}^n$. In the general fts, there are many topological properties, such as the closure, the interior, the boundary, the core, the fringe, the frontier, the internal, the internal fringe, the outer, the exterior and so on. In $\widetilde{R}^n$ the fringe of a fuzzy set is equal to the boundary, the internal fringe is equal to the internal boundary, and the frontier is equal to the frontier of the boundary. We adopt the topological properties of a fuzzy *n*-cell in $\widetilde{R}^n$. That is, the *closure* of a fuzzy *n*-cell $\widetilde{\widetilde{e}}_n$ is just the fuzzy *n*-cell; the *interior* of a fuzzy *n*-cell is the interior of the fuzzy *n*-cell in $\widetilde{R}^n$, which is a fuzzy open *n*-cell $\widetilde{e}_n$; the *core* $\widetilde{\widetilde{e}}_n^{\oplus}$ of a fuzzy *n*-cell is the crisp subset of the interior of the fuzzy *n*-cell in $\widetilde{R}^n$; the *boundary* (or *fringe*) $\partial\widetilde{\widetilde{e}}_n$ is the difference between the fuzzy *n*-cell and the core of the fuzzy *n*-cell; and the *frontier* $\partial^e\widetilde{\widetilde{e}}_n$ of a fuzzy *n*-cell is the subset of the *n*-cell in $\widetilde{R}^n$ where the membership value of the *n*-cell is greater than the membership value of the interior of the *n*-cell in $\widetilde{R}^n$. The difference between the boundary and the frontier is the *internal boundary* (or *internal fringe*) $\partial^i\widetilde{\widetilde{e}}_n$ of a fuzzy *n*-cell.

It should be noted that these topological properties are defined based on the fact that a fuzzy *n*-cell has the same dimension as the fuzzy Euclidean space. For example, the topological properties of a 2-cell are defined based on the fact that the dimension 2 of the 2-cell is equal to the dimension of $\widetilde{R}^2$. This is because these topological properties are different in different fuzzy Euclidean spaces. For example, the interior of a 2-cell is a fuzzy open disk in $\widetilde{R}^2$, but the interior of a 2-cell is empty in $\widetilde{R}^3$.

In a crisp cell structure, the boundary of a cell has always lower dimension than that of the cell. For example, the boundary of a 2-cell is always one-dimensional. The boundary of a 1-cell is always composed of two end points, which is zero-dimensional. In a fuzzy cell structure, the boundary of an *n*-cell may have the same dimension as the *n*-cell. Figure 5.6 shows some forms of a 1-cell and 2-cell. Their topological properties are illustrated in Figure 5.7.

Figure 5.6 Different forms of a fuzzy 1-cell and 2-cell

5.3.2 Fuzzy cell complex

In a crisp cell complex, each *n*-cell is attached onto *p*-cells where $p < n$, and the dimension of the boundary of a crisp *n*-cell is always less than the dimension of the *n*-cell and that for fuzzy cells this does not hold. The boundary of a fuzzy *n*-cell may have the same dimension as the fuzzy *n*-cell. In order to define a fuzzy cell complex based on fuzzy cells, we define the external frontier of a fuzzy *n*-cell. The *external frontier* $\partial^{ex}\widetilde{e}_n$ of a fuzzy *n*-cell is the subset of the frontier for which the distance $d(x_a, y_b) = 1$ around the center *x* in $\widetilde{R}^n$. The external frontier is one-dimension less than a fuzzy *n*-cell since it is homeomorphic to the boundary of the support of a fuzzy *n*-cell in $\widetilde{R}^n$. Figure 5.7 shows the external frontier and other topological properties.

In general, the frontier, the external frontier and the boundary are different for a fuzzy cell. If a fuzzy *n*-cell is crisp, then the frontier, the external frontier and the boundary are the same. The internal boundary of a crisp *n*-cell is empty. Figure 5.7 shows these properties of the fuzzy 1-cell and fuzzy 2-cell (the last two figures) described in Figure 5.6.

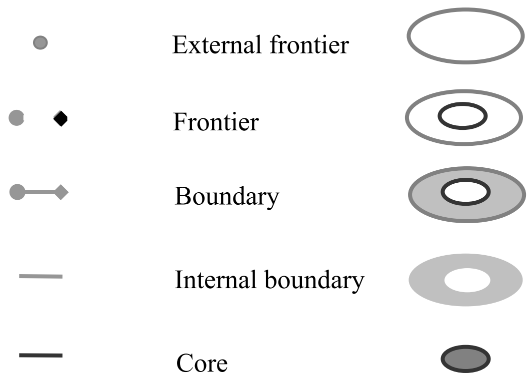


Figure 5.7 Topological properties of the fuzzy 1- and 2-cells in Figure 5.6

Definition 5.1 A *fuzzy cell complex X is a collection of disjoint fuzzy open cells* $\widetilde{e}_i^n$

whose union is X, such that:

(1) *The support of X is an induced space of a crisp Hausdorff topological space*;

(2) *For each open fuzzy n-cell* $\tilde{e}_i^n$ *of the collection, there exists a continuous mapping* $f_i : \mathrm{supp}(D^n) \to \mathrm{supp}(X)$ *that maps the interior of a fuzzy closed disk* $(D^o)^n$ *(fuzzy) homeomorphically onto* $\tilde{e}_i^n$ *and carries* $\partial^{ex}(D^n)$ *into a finite union of fuzzy open cells, each of dimension less than n*;

(3) *A set A is closed in X if* $A \cap \tilde{\bar{e}}_i^n$ *is closed in* $\bar{e}_i^n$ *for each i.*

A *finite fuzzy cell complex X* is a fuzzy cell complex such that the cells are finite. Call each 0,1,…,*n*-cell in *X* a *face* of *X*. A *proper face* is any 0,1,…,(*n*-1)-cell. A fuzzy cell complex has the structure that each fuzzy *n*-cell is attached onto a *p*-cell where $p < n$ along the external frontier of each *n*-cell (*n*>0). Therefore a proper face is in the external frontier of fuzzy 1,2,…,*n*-cells. Since an *n*-cell has an *n*-sub-cell structure, a face also holds this structure.

We simply call *X* a fuzzy complex. A subset of a complex is called a *fuzzy sub-complex* if it is a subset of the complex and still a fuzzy complex. A finite fuzzy cell complex *X* can be formed in terms of faces. *X* is called a fuzzy cell complex if (1) every face of a fuzzy cell of *X* is in *X*, and (2) the intersection of two cells is either empty or a common face of both fuzzy cells.

5.4 Modeling fuzzy spatial objects

5.4.1 Definition of fuzzy spatial objects

In order to represent fuzzy spatial objects, we limit a fuzzy complex *X* to be finite and two-dimensional. Then *X* can be imbedded in $\tilde{R}^2$. The topology of support of *X* is then equivalent to the topology of support of *X* that is inherited from $\tilde{R}^2$. We can adopt the concept of all the topological properties of a sub-complex in $\tilde{R}^2$. We can define fuzzy spatial objects to be a fuzzy complex based on these topological properties.

Definition 5.2 *Let A be a sub-complex composed of fuzzy 2-cells* $\bar{e}_i$ *of finite fuzzy complex X. A is called a simple fuzzy region in* $\tilde{R}^2$ *such that the union of fuzzy cells* $\bigcup \bar{e}_i$ *meets the following conditions in* $\tilde{R}^2$:

(1) *It is a non-empty proper double-connected closed set;*

(2) *The interior, the core and the outer are double-connected regular open;*

(3) *The support is equal to the support of the closure of the interior;*

(4) *The boundary is double-connected and the internal boundary is a double-connected open set;*

(5) *The frontier is a closed set.*

Actually, in $\tilde{R}^2$ condition (5) is trivial according to the definition of a fuzzy complex. It is also trivial that the core and the outer are regular open in $\tilde{R}^2$. The internal boundary is open also in $\widetilde{R}^2$. Therefore these conditions can be removed for a simple fuzzy region in $\widetilde{R}^2$. We just quote it for consistency with the formal definition of a simple fuzzy region. In $\widetilde{R}^2$ the fringe is equal to the boundary. Therefore, the fringe and the internal fringe are changed into the boundary and the internal boundary as usual.

Definition 5.3 *Let A be a sub-complex composed of fuzzy 2-cells of finite fuzzy complex X. A is called a fuzzy region in* $\widetilde{R}^2$*, such that the union of fuzzy cells* $\bigcup \overline{e}_i$ *meets the following conditions in* $\widetilde{R}^2$*:*

(1) *It is a non-empty proper double-connected closed set;*
(2) *The interior is a double-connected regular open set;*
(3) *The support is equal to the support of the closure of the interior;*
(4) *The core, the internal boundary and the outer are respectively a collection of subsets such that these subsets are mutually disjoint and every subset is double-connected.*

Definition 5.4 *Let A be a sub-complex composed of fuzzy 1-cells of finite fuzzy complex X. A is called a simple fuzzy line in* $\widetilde{R}^2$ *if the union of fuzzy cells* $\bigcup \overline{e}_i$ *meets the following conditions:*

(1) *It is a non-empty double-connected closed set in* $\widetilde{R}^2$*;*
(2) *Its support is not self-intersecting and does not form a loop in* $\widetilde{R}^2$, and *is equal to the support of the closure of the interior in* $\widetilde{R}$*;*
(3) *The interior is a non-empty double-connected regular open set in* $\widetilde{R}$.
(4) *The internal boundary is a collection of two subsets such that they are mutually disjoint and every subset is double-connected in* $\widetilde{R}$;
(5) *The core is double-connected in* $\widetilde{R}$;

Definition 5.5 *Let A be a sub-complex composed of fuzzy 1-cells of finite fuzzy complex X. A is called a fuzzy line in* $\widetilde{R}^2$ *if the union of fuzzy cells* $\bigcup \overline{e}_i$ *meets the following conditions:*

(1) *It is a non-empty double-connected closed set in* $\widetilde{R}^2$;
(2) *Its support is not self-intersecting and does not form a loop in* $\widetilde{R}^2$*, and is equal to the support of the closure of the interior in* $\widetilde{R}$;
(3) *The interior is a non-empty double-connected regular open set in* $\widetilde{R}$;
(4) *The core and the internal boundary are respectively a collection of subsets such that the subsets are mutually disjoint and every subset is double-connected in* $\widetilde{R}$.

Definition 5.6 *A fuzzy point is a fuzzy 0-cell in* $\widetilde{R}^2$.

A simple fuzzy region, a fuzzy region, a simple fuzzy line and a fuzzy line are illustrated in Figure 5.8.

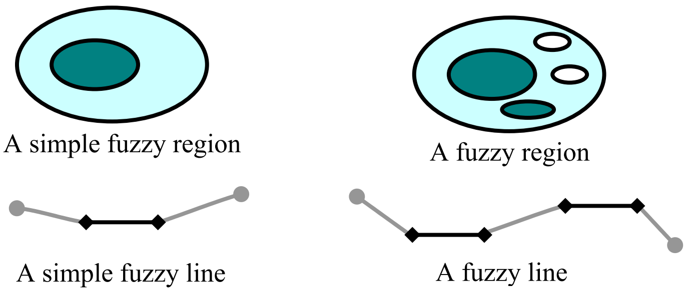


Figure 5.8 Simple fuzzy region, fuzzy region, simple fuzzy line and fuzzy line

5.4.2 Representation of fuzzy spatial objects

A fuzzy spatial object abstracted from the real world can be easily expressed by a fuzzy complex. For example, a mountain can be represented by a fuzzy complex. This complex has two cores in $\widetilde{R}^2$ (Figure 5.9).

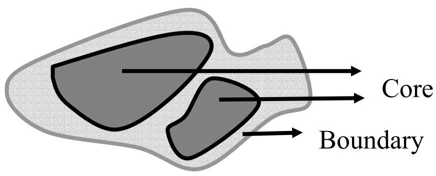


Figure 5.9 Representation of a fuzzy spatial object

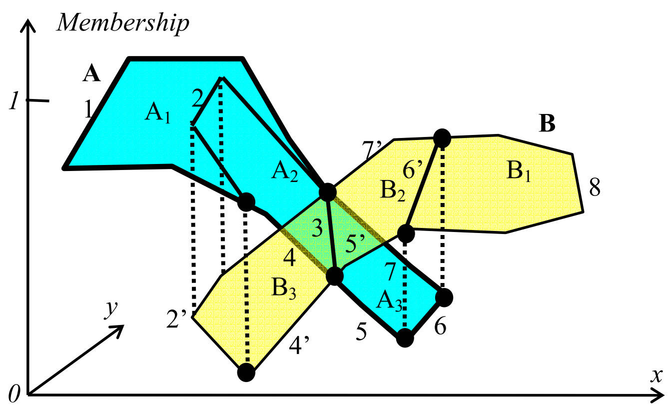


Figure 5.10 Representation of fuzzy spatial objects

When more fuzzy objects are involved, we can identify cells according to the cell complex structure. For example there are two fuzzy objects: forest (A) and grassland (B), showing in Figure 5.10. In total there are six fuzzy 2-cells ($A_1, A_2, A_3, B_1, B_2, B_3$). 13 fuzzy 1-cells (1, 2, 3, 4, 5, 6, 7, 8, 2', 4', 5', 6', 7') and eight fuzzy 0-cells. The union of A_1, A_2, A_3 and the union of B_1, B_2, B_3 constitute two fuzzy regions representing the forest and the grassland. B_3 is a common 2-cell of A_2, B_3. It is actually a 2-sub-cell of A_2. 1-cell 2 is a common face of A_1, A_2. 1-cell 2' is a common face of A_1, B_3.

5.5 Topological relations between fuzzy spatial objects

5.5.1 Topological relations between simple fuzzy regions

The topological relations can be directly identified based on the 3*3-intersection matrix (1) in Chapter 4, since the definition of a simple fuzzy region is directly adopted for a simple fuzzy region in $\widetilde{R}^2$. Forty-four (44) topological relations can be realized between two simple regions when the empty/non-empty contents are applied as the topological invariants of the intersections. These relations can be further decomposed into 77 relations if the comparatives "⊆","⊇","=" and "≠" between the topological parts of two real fuzzy simple regions are adopted.

5.5.2 Topological relations between simple fuzzy lines

In order to identify the topological relations between two simple fuzzy lines in $\widetilde{R}^2$ by using the 3*3-intersection matrix (1) depicted in Chapter 4, we can replace the core, the fringe and the outer of a simple fuzzy line in $\widetilde{R}^2$ by the core, the fringe and the outer in $\widetilde{R}$. This is because the core, the fringe and the outer of a simple line in $\widetilde{R}$ are still the topological properties in $\widetilde{R}^2$, since the core and the outer of a simple line in $\widetilde{R}$ are crisp subsets in $\widetilde{R}^2$, and the fringe is the intersection of a simple line with the set that $\widetilde{R}^2$ intersects with the complement of the union of the above crisp sets in $\widetilde{R}^2$.

Limitations (1) to (4) between two simple fuzzy regions (listed in Section 4.5.3) still hold between two simple fuzzy lines. Furthermore, there are other limitations between two simple fuzzy lines:

(1) If the boundary of *A* does not intersect the boundary of *B*, and the core of *A* intersects the core and boundary of *B*, then the core of *A* must intersect the exterior of *B*, and vice versa;
(2) If the core of *A* does not intersect the boundary of *B*, and the boundary of *A* does not intersect the core of *B*, and if both cores intersect each other and both boundaries intersect each other, then either both cores intersect the exterior of the other, or both cores cannot intersect the exterior of the other, and vice versa;
(3) If both cores do not intersect each other, and both boundaries do not intersect each other, then their cores intersect the exterior of the opposite;
(4) If both boundaries do not intersect the exterior of the opposite, then either both cores intersect the exterior of the other, or both cores cannot intersect the

exterior of the other;

(5) If the core of *A* intersects all parts except the boundary of *B*, then the exterior must intersect with the core of *B*, and vice versa.

By using the 3*3-intersection matrix (3) in Section 4.5.1, 97 relations can be identified between two simple fuzzy lines (Table 5.1).

Table 5.1 Ninety-seven (97) relations between two simple fuzzy lines

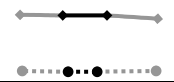
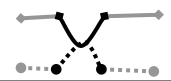
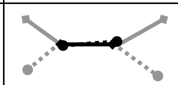
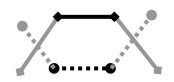
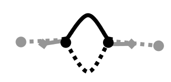
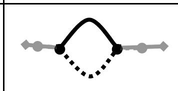
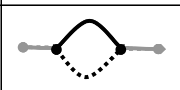
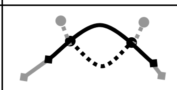
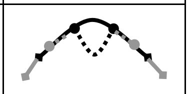
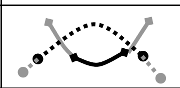
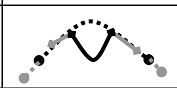
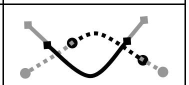
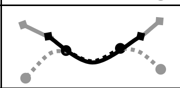
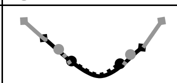
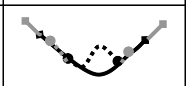
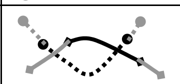
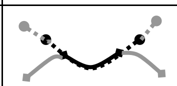
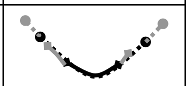
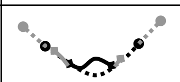
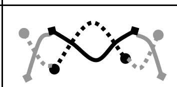
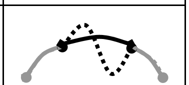
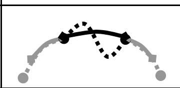
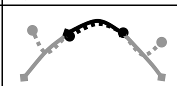
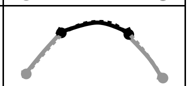
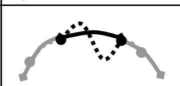
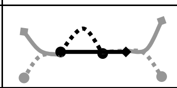
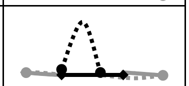
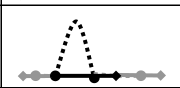
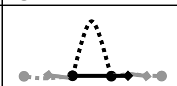
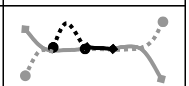
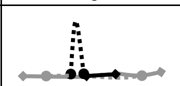
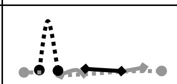
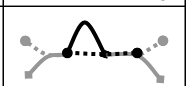
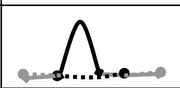
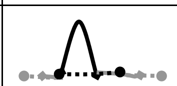
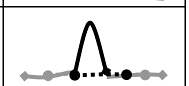
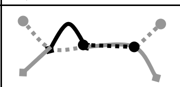
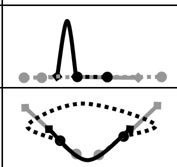
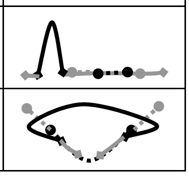
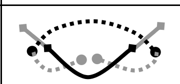
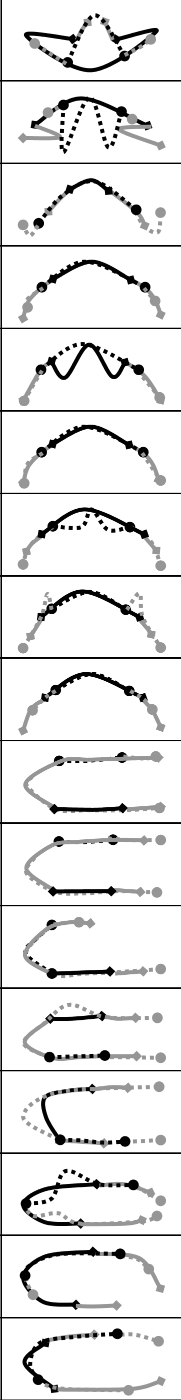
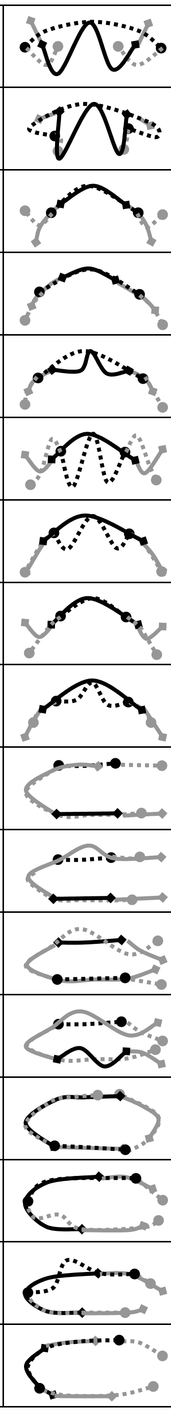
Matrix	Illustration	Matrix	Illustration	Matrix	Illustration
(1) $\begin{bmatrix}0&0&1\\0&0&1\\1&1&1\end{bmatrix}$		(2) $\begin{bmatrix}1&0&1\\0&0&1\\1&1&1\end{bmatrix}$		(3) $\begin{bmatrix}1&0&0\\0&0&1\\0&1&1\end{bmatrix}$	
(4) $\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$		(5) $\begin{bmatrix}0&0&1\\0&1&0\\1&1&1\end{bmatrix}$		(6) $\begin{bmatrix}0&0&1\\0&1&1\\1&0&1\end{bmatrix}$	
(7) $\begin{bmatrix}0&0&1\\0&1&0\\1&0&1\end{bmatrix}$		(8) $\begin{bmatrix}0&1&1\\0&0&1\\1&1&1\end{bmatrix}$		(9) $\begin{bmatrix}0&1&1\\0&0&1\\1&0&1\end{bmatrix}$	
(10) $\begin{bmatrix}0&0&1\\1&0&1\\1&1&1\end{bmatrix}$		(11) $\begin{bmatrix}0&0&1\\1&0&0\\1&1&1\end{bmatrix}$		(12) $\begin{bmatrix}1&1&1\\0&0&1\\1&1&1\end{bmatrix}$	
(13) $\begin{bmatrix}1&1&1\\0&0&1\\0&1&1\end{bmatrix}$		(14) $\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}$		(15) $\begin{bmatrix}1&1&1\\0&0&1\\1&0&1\end{bmatrix}$	
(16) $\begin{bmatrix}1&0&1\\1&0&1\\1&1&1\end{bmatrix}$		(17) $\begin{bmatrix}1&0&0\\1&0&1\\1&1&1\end{bmatrix}$		(18) $\begin{bmatrix}1&0&0\\1&0&0\\1&1&1\end{bmatrix}$	
(19) $\begin{bmatrix}1&0&1\\1&0&0\\1&1&1\end{bmatrix}$		(20) $\begin{bmatrix}1&0&1\\0&1&1\\1&1&1\end{bmatrix}$		(21) $\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$	
(22) $\begin{bmatrix}1&0&1\\0&1&0\\1&1&1\end{bmatrix}$		(23) $\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$		(24) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	
(25) $\begin{bmatrix}1&0&1\\0&1&1\\1&0&1\end{bmatrix}$		(26) $\begin{bmatrix}0&1&1\\0&1&1\\1&1&1\end{bmatrix}$		(27) $\begin{bmatrix}0&1&1\\0&1&0\\1&0&1\end{bmatrix}$	
(28) $\begin{bmatrix}0&1&1\\0&1&1\\1&0&1\end{bmatrix}$		(29) $\begin{bmatrix}0&1&1\\0&1&0\\1&1&1\end{bmatrix}$		(30) $\begin{bmatrix}0&1&0\\0&1&1\\1&1&1\end{bmatrix}$	
(31) $\begin{bmatrix}0&1&0\\0&1&1\\1&0&1\end{bmatrix}$		(32) $\begin{bmatrix}0&1&0\\0&1&0\\1&1&1\end{bmatrix}$		(33) $\begin{bmatrix}0&0&1\\1&1&1\\1&1&1\end{bmatrix}$	
(34) $\begin{bmatrix}0&0&1\\1&1&0\\1&0&1\end{bmatrix}$		(35) $\begin{bmatrix}0&0&1\\1&1&0\\1&1&1\end{bmatrix}$		(36) $\begin{bmatrix}0&0&1\\1&1&1\\1&0&1\end{bmatrix}$	
(37) $\begin{bmatrix}0&0&1\\1&1&1\\0&1&1\end{bmatrix}$		(38) $\begin{bmatrix}0&0&1\\1&1&0\\0&1&1\end{bmatrix}$		(39) $\begin{bmatrix}0&0&1\\1&1&1\\0&0&1\end{bmatrix}$	
(40) $\begin{bmatrix}0&1&1\\1&0&1\\1&1&1\end{bmatrix}$		(41) $\begin{bmatrix}0&1&1\\1&0&1\\1&0&1\end{bmatrix}$		(42) $\begin{bmatrix}0&1&1\\1&0&0\\1&1&1\end{bmatrix}$	

Table 5.1 cont'd

(43) $\begin{bmatrix}0&1&1\\1&0&0\\1&0&1\end{bmatrix}$	(44) $\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}$	(45) $\begin{bmatrix}1&1&1\\1&0&0\\1&1&1\end{bmatrix}$
(46) $\begin{bmatrix}1&1&1\\1&0&1\\1&0&1\end{bmatrix}$	(47) $\begin{bmatrix}1&1&1\\1&0&0\\1&0&1\end{bmatrix}$	(48) $\begin{bmatrix}1&0&0\\1&1&1\\1&1&1\end{bmatrix}$
(49) $\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$	(50) $\begin{bmatrix}1&0&0\\1&1&1\\0&1&1\end{bmatrix}$	(51) $\begin{bmatrix}1&0&0\\1&1&1\\1&0&1\end{bmatrix}$
(52) $\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix}$	(53) $\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$	(54) $\begin{bmatrix}1&0&1\\1&1&1\\1&0&1\end{bmatrix}$
(55) $\begin{bmatrix}1&0&1\\1&1&0\\1&0&1\end{bmatrix}$	(56) $\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}$	(57) $\begin{bmatrix}1&0&1\\1&1&1\\1&1&1\end{bmatrix}$
(58) $\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$	(59) $\begin{bmatrix}1&1&1\\0&1&1\\1&1&1\end{bmatrix}$	(60) $\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$
(61) $\begin{bmatrix}1&1&1\\0&1&0\\1&1&1\end{bmatrix}$	(62) $\begin{bmatrix}1&1&1\\0&1&0\\1&0&1\end{bmatrix}$	(63) $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$
(64) $\begin{bmatrix}1&1&1\\0&1&0\\0&1&1\end{bmatrix}$	(65) $\begin{bmatrix}1&1&0\\0&1&1\\0&1&1\end{bmatrix}$	(66) $\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix}$
(67) $\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$	(68) $\begin{bmatrix}1&1&1\\0&1&1\\1&0&1\end{bmatrix}$	(69) $\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$
(70) $\begin{bmatrix}0&1&0\\1&1&0\\0&0&1\end{bmatrix}$	(71) $\begin{bmatrix}0&1&0\\1&1&1\\1&1&1\end{bmatrix}$	(72) $\begin{bmatrix}0&1&0\\1&1&1\\0&0&1\end{bmatrix}$
(73) $\begin{bmatrix}0&1&0\\1&1&0\\0&1&1\end{bmatrix}$	(74) $\begin{bmatrix}0&1&0\\1&1&1\\1&0&1\end{bmatrix}$	(75) $\begin{bmatrix}0&1&0\\1&1&0\\1&1&1\end{bmatrix}$
(76) $\begin{bmatrix}0&1&0\\1&1&1\\0&1&1\end{bmatrix}$	(77) $\begin{bmatrix}0&1&1\\1&1&1\\0&1&1\end{bmatrix}$	(78) $\begin{bmatrix}0&1&1\\1&1&1\\0&0&1\end{bmatrix}$
(79) $\begin{bmatrix}0&1&1\\1&1&0\\0&1&1\end{bmatrix}$	(80) $\begin{bmatrix}0&1&1\\1&1&1\\1&1&1\end{bmatrix}$	(81) $\begin{bmatrix}0&1&1\\1&1&1\\1&0&1\end{bmatrix}$
(82) $\begin{bmatrix}0&1&1\\1&1&0\\1&1&1\end{bmatrix}$	(83) $\begin{bmatrix}0&1&1\\1&1&0\\1&0&1\end{bmatrix}$	(84) $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$
(85) $\begin{bmatrix}1&1&1\\1&1&0\\1&1&1\end{bmatrix}$	(86) $\begin{bmatrix}1&1&1\\1&1&0\\0&1&1\end{bmatrix}$	(87) $\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}$
(88) $\begin{bmatrix}1&1&1\\1&1&1\\0&0&1\end{bmatrix}$	(89) $\begin{bmatrix}1&1&1\\1&1&1\\1&0&1\end{bmatrix}$	(90) $\begin{bmatrix}1&1&1\\1&1&1\\0&1&1\end{bmatrix}$
(91) $\begin{bmatrix}1&1&0\\1&1&1\\1&1&1\end{bmatrix}$	(92) $\begin{bmatrix}1&1&0\\1&1&0\\1&1&1\end{bmatrix}$	(93) $\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}$





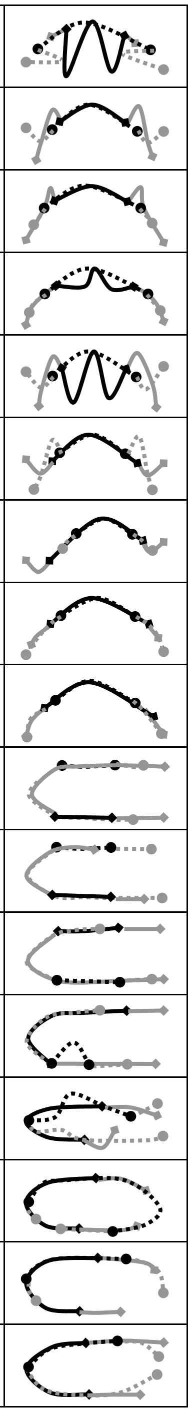
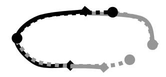
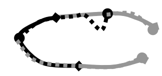
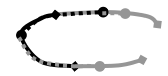
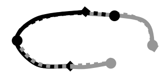


Table 5.1 cont'd

(94) $\begin{bmatrix}1&1&0\\1&1&0\\0&1&1\end{bmatrix}$		(95) $\begin{bmatrix}1&1&0\\1&1&1\\1&0&1\end{bmatrix}$		(96) $\begin{bmatrix}1&1&0\\1&1&1\\0&0&1\end{bmatrix}$	
(97) $\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$					

5.5.3 Topological relations between simple fuzzy region and simple fuzzy line

Besides limitations (1) to (4) in Section 4.5.3, there are five more limitations between a simple fuzzy region and a simple fuzzy line.

(1) The core and the boundary of a fuzzy simple region always intersect with the exterior of a simple fuzzy line;
(2) If the core of a simple fuzzy line is a subset of the support of the boundary of a simple fuzzy region, then the boundaries must intersect each other;
(3) If both boundaries do not intersect each other, and the core of a simple fuzzy line is a subset of the core of a simple fuzzy region, then the closure of the simple fuzzy line is a subset of the core of a simple fuzzy region;
(4) If both cores do not intersect each other, and the boundary of a simple fuzzy line intersects the core of a simple fuzzy region, then it also intersects the boundary of the simple fuzzy region;
(5) If both cores intersect each other, and the core of a simple fuzzy region does not intersect the boundary of a simple fuzzy region, then it does not intersect the exterior of the simple fuzzy region.

Thirty (30) relations between a simple fuzzy region and a simple fuzzy line are identified based on these limitations (Table 5.2).

Table 5.2 Thirty (30) relations between a simple fuzzy region and a simple fuzzy line

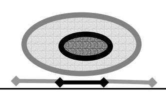
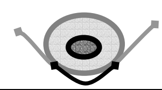
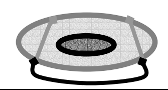
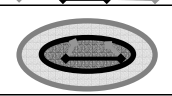
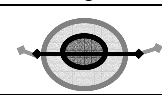
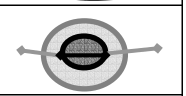
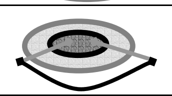
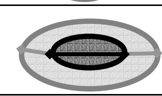
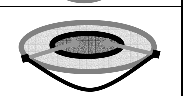
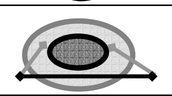
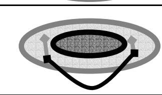
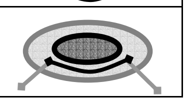
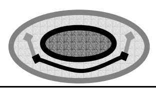
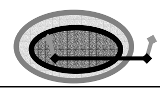
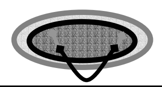
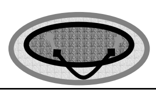
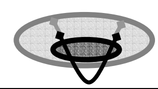
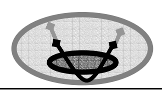
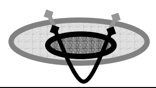
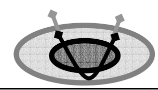
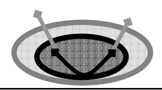
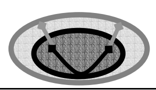
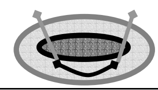
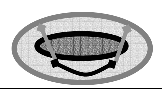
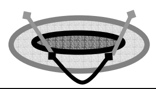
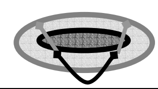
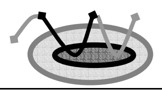
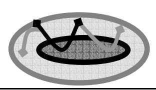
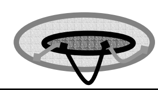
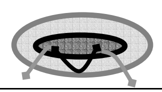
Matrix	Illustration	Matrix	Illustration	Matrix	Illustration
(1) $\begin{bmatrix}0&0&1\\0&0&1\\1&1&1\end{bmatrix}$		(2) $\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$		(3) $\begin{bmatrix}0&0&1\\0&1&1\\1&0&1\end{bmatrix}$	
(4) $\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}$		(5) $\begin{bmatrix}1&0&1\\1&0&1\\1&1&1\end{bmatrix}$		(6) $\begin{bmatrix}1&0&1\\0&1&1\\0&1&1\end{bmatrix}$	
(7) $\begin{bmatrix}0&1&1\\0&1&1\\1&1&1\end{bmatrix}$		(8) $\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}$		(9) $\begin{bmatrix}0&1&1\\0&1&1\\1&0&1\end{bmatrix}$	
(10) $\begin{bmatrix}0&0&1\\1&1&1\\1&1&1\end{bmatrix}$		(11) $\begin{bmatrix}0&0&1\\1&1&1\\1&0&1\end{bmatrix}$		(12) $\begin{bmatrix}0&0&1\\1&1&1\\0&1&1\end{bmatrix}$	

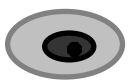
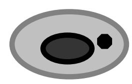
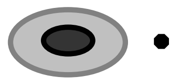
Table 5.2 cont'd

(13) $\begin{bmatrix}0&0&1\\1&1&1\\0&0&1\end{bmatrix}$		(14) $\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}$		(15) $\begin{bmatrix}1&1&1\\1&0&1\\1&0&1\end{bmatrix}$	
(16) $\begin{bmatrix}1&1&1\\1&0&1\\0&0&1\end{bmatrix}$		(17) $\begin{bmatrix}1&0&1\\1&1&1\\1&0&1\end{bmatrix}$		(18) $\begin{bmatrix}1&0&1\\1&1&1\\0&0&1\end{bmatrix}$	
(19) $\begin{bmatrix}1&0&1\\1&1&1\\1&1&1\end{bmatrix}$		(20) $\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix}$		(21) $\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$	
(22) $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$		(23) $\begin{bmatrix}0&1&1\\1&1&1\\0&1&1\end{bmatrix}$		(24) $\begin{bmatrix}0&1&1\\1&1&1\\0&0&1\end{bmatrix}$	
(25) $\begin{bmatrix}0&1&1\\1&1&1\\1&1&1\end{bmatrix}$		(26) $\begin{bmatrix}0&1&1\\1&1&1\\1&0&1\end{bmatrix}$		(27) $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	
(28) $\begin{bmatrix}1&1&1\\1&1&1\\0&0&1\end{bmatrix}$		(29) $\begin{bmatrix}1&1&1\\1&1&1\\1&0&1\end{bmatrix}$		(30) $\begin{bmatrix}1&1&1\\1&1&1\\0&1&1\end{bmatrix}$	

5.5.4 Topological relations between a fuzzy point and a fuzzy line/fuzzy region

Since the boundary of a fuzzy point is empty, there is a strong limitation between a fuzzy point and a simple fuzzy line or a simple fuzzy region. That is, a fuzzy point is contained in only one support of parts of the line/region. There are three relations between a fuzzy point and a simple fuzzy line and three relations between a fuzzy point and a simple fuzzy region (Table 5.3).

Table 5.3 Three relations between a fuzzy point and a simple fuzzy line/a simple fuzzy region

Matrix	Illustration	Matrix	Illustration	Matrix	Illustration
(1) $\begin{bmatrix}1&0&1\\0&0&1\\0&0&1\end{bmatrix}$		(2) $\begin{bmatrix}0&0&1\\1&0&1\\0&0&1\end{bmatrix}$		(3) $\begin{bmatrix}0&0&1\\0&0&1\\1&0&1\end{bmatrix}$	
(1) $\begin{bmatrix}1&0&1\\0&0&1\\0&0&1\end{bmatrix}$		(2) $\begin{bmatrix}0&0&1\\1&0&1\\0&0&1\end{bmatrix}$		(3) $\begin{bmatrix}0&0&1\\0&0&1\\1&0&1\end{bmatrix}$	

5.6 Conclusions and discussions

This chapter proposes a theoretic framework for modeling fuzzy spatial objects based on algebraic topology. The framework is a fuzzy cell complex structure that is constructed based on fuzzy cells.

A fuzzy cell is defined as a subset whose space is homeomorphic to a fuzzy open interval in the fuzzy Euclidean space. It is a natural extension of a crisp cell. The topological properties of a fuzzy cell such as the core, the boundary, the internal boundary are defined. A fuzzy cell complex can then be built based on fuzzy cells. It is an extension of a crisp cell complex.

The structure of a fuzzy cell complex can be used to model fuzzy spatial objects. The formal definitions of fuzzy points, simple fuzzy lines, fuzzy lines, simple fuzzy regions and fuzzy regions are proposed based on the fuzzy cell complex in the fuzzy Euclidean space.

The topological relations between simple fuzzy regions, simple fuzzy lines and fuzzy points are investigated since they are basic to spatial data modeling. The idea for the identification is based on the 3*3-intersection matrix, which results from fuzzy topology. Forty-four (44) relations between two simple fuzzy regions, 97 relations between two simple fuzzy lines, 30 relations between a simple fuzzy region and a simple fuzzy line, three relations between a fuzzy point and a simple fuzzy region, and three relations between a fuzzy point and a simple fuzzy line are derived. It can be easily perceived that the relations between fuzzy spatial objects are also just an extension of those between crisp spatial objects. If all fuzzy objects are crisp, the relations between these objects turn out to be crisp relations, just as the relations identified by Egenhofer and Herring (1990a). Therefore the framework of fuzzy cell complex is compatible with data models for crisp spatial objects.

The above relations are all based on the 3*3-intersection matrix. More topological relations can be identified if the 4*4-intersection matrix is adopted. One hundred and fifty-two (152) relations can be identified between two simple fuzzy regions in $\widetilde{R}^2$. This is useful when more topological relations between two fuzzy spatial objects should be differentiated. These relations can be refined if the comparative invariants are adopted. For example, by using four comparatives in the 3*3-intersection matrix, 77 relations can be identified between two real simple fuzzy regions. The same method can be adopted for the comparison between fuzzy lines and features of other dimensions.

Chapter Six

Generating Fuzzy Land Cover Objects

6.1 Introduction

In previous chapters fuzzy spatial objects have been formally defined based on fuzzy topology, and the topological relations between fuzzy spatial objects have also been identified based on the intersection matrices. These frameworks should be applied to solve the practical problems in reality. Starting with this chapter, some practical issues on fuzzy spatial objects will be investigated.

The first practical issue is how to generate fuzzy spatial objects. In conventional GIS, crisp spatial objects can be derived by many methods: digitizing, for instance manual digitizing and on-screen digitizing (automated or semi-automated); inputting from other data source, for instance GPS data; adopting processed data such as classification results of satellite images, etc. In principle, a fuzzy spatial object can also be generated by the above method. The only difference in generating fuzzy spatial objects is that we have to derive the membership values or membership functions of fuzzy spatial objects whereas this is not necessary for crisp spatial objects. When spatial objects are not fuzzy (for instance administrative boundaries), it is not necessary to model them in fuzzy mode. When there is some fuzziness in a spatial object, whether fuzzy or crisp, a spatial object is decided by the requirement of the application. For example, most land cover objects such as grassland contain spatial fuzziness, since the definition is not clear. If the application just addresses the area size, then the crisp representation of land covers is enough. However, if the application focuses on the changes, especially on gradual changes, it is better to represent the land cover objects that have fuzzy characteristics by fuzzy land cover objects.

The key to generating fuzzy spatial objects is to deriving the membership values. In general, there are two kinds of methods: active and passive. In the active method, membership function and values are derived by experts or based on some knowledge. The passive method calculates the membership functions according to the data itself. Cheng *et al.* (2001) discussed the general processes of identifying fuzzy spatial objects.

They also proposed three models to form fuzzy objects using active and passive methods. However, the passive method, which was applied in TM image classification, needs more discussions, since the extent of fuzzy spatial objects covers too broad an area.

This chapter discusses a composite method for forming fuzzy land cover objects (Tang and Kainz, 2003). Section 6.2 introduces the general procedure for forming fuzzy spatial objects by using the passive method. In Section 6.3, a composite approach is proposed for computing the membership values for the fuzzy land cover objects. Section 6.4 shows the situation of the test area. Section 6.5 discusses the generation procedures, which involve many steps such as classification, fuzzy convolution, rule-based processing etc. Section 6.6 discusses the accuracy of fuzzy land cover objects. The results show that the method is applicable for the generation of data-oriented fuzzy spatial objects. Finally the conclusions and discussions are summarized.

6.2 General procedure for generating fuzzy spatial objects

6.2.1 Procedure for forming fuzzy spatial objects

The general procedures for identifying fuzzy spatial objects consist of three steps (Figure 6.1) (Cheng *et al.* 1997): analysis of fuzzy type of spatial objects, computation of membership values and evaluation of the accuracy of fuzzy spatial objects.

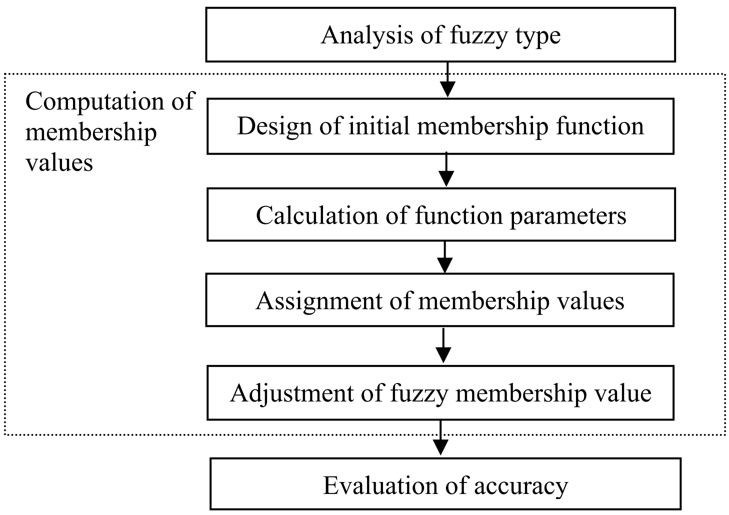


Figure 6.1 General procedure for forming fuzzy spatial objects

6.2.2 Analysis of fuzzy type

Understanding fuzzy type is the starting point for generating fuzzy spatial objects. At this step, the aspects that cause the fuzziness of spatial objects should be interpreted so as to ascertain whether they are related to certain applications.

In general two types of fuzziness exist in spatial objects: spatial extent fuzziness and thematic fuzziness (object definition). The thematic fuzziness exists when we cannot clearly define an object. For example, when we define a land cover type, although we try to determine each land cover clearly, the fuzziness is still inevitable in each class. When we browse the definition of forest in the USGS land cover standard (Anderson *et al*. 1976), we find it is defined as an area characterized by tree cover (natural or semi-natural woody vegetation, generally greater than 6 m tall); tree canopy accounts for 25 to 100% of the cover. In this definition, "natural", "semi-natural" and "generally greater than 6 m tall" are fuzzy terms. Nor is the area size specified in the definition. Another kind of fuzziness is the spatial extent fuzziness. Sometimes, we can clearly define an object, but we cannot clearly obtain it. When a TM image is classified into land cover classes such as grassland, we will immediately find that some pixels are a mixture of grassland with some trees or dry land; some pixels have some grassland at one side and other land covers at the other side; and some pixels contain both of the above cases.

In some applications, the location can be measured precisely, leaving the object definition fuzzy. In other applications, the definition is clear but the location cannot be measured precisely. In some cases, both definitions and locations contain fuzziness. In general, we should bear in mind which fuzziness is mainly concerned in the applications.

6.2.3 Computation of membership values

The computation of membership values should be done for all fuzzy spatial objects. Generally it consists of the following steps:

(1) Design of initial membership functions

After understanding which object fuzziness is concerned, the initial membership functions should be designed. The design is usually done by selecting one of the existing membership functions, such as triangular, trapezoidal, bell-shaped or Gaussian distribution, based on the fitness of the fuzziness of spatial objects with these functions. For example, if we subdivide the human stature into three fuzzy classes *short, middle and tall*, the trapezoidal membership functions (Figure 6.2) can be adopted as initial functions for short, middle and tall.

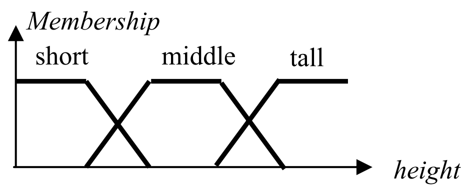


Figure 6.2 Membership functions for human stature *short, middle and tall*

(2) Calculation of parameters

The methods of parameter calculation can be generally classified into two categories: active and passive. Usually the passive method will be adopted. The active method will be selected when the passive method cannot be adopted for a certain application.

(3) Assignment of fuzzy membership values

After the membership function and its parameters are determined, the membership values can be calculated at each location of the spatial objects. Usually, each location is described by a pixel so that the membership values can be recorded at each pixel. Otherwise, the membership value has to be calculated at run-time in GIS models. However, this way is seldom adopted since it has to store the functions in the data model.

(4) Adjustment of membership values

In many cases, the membership functions and values have to be adjusted to meet the factual situation and the application requirements. Because of the complexity of spatial features and problems in data sampling, there could be errors in the membership functions or values. In some situations, although the membership values can reflect the factual situation, they are too complicated for applications. For example, the extents of spatial objects are too small. Thus the analyses are too time-consuming and the visualization is very poor. To minimize the above side effects, the membership values should be adjusted to facilitate the analysis.

6.2.4 Evaluation of accuracy

The evaluation of accuracy is the final step in forming fuzzy spatial objects. The evaluation can be done on two levels. The first level checks the errors in classification, that is, whether the object type is correct or not. The second level verifies the degree of fitness of the fuzziness with the factual situation. Normally, field survey should be done to verify both accuracies of fuzzy spatial objects.

6.3 Method for generating fuzzy land cover objects

6.3.1 Fuzziness in land cover objects

The importance of land cover needs no more explanation, since it plays a fundamental role in many fields such as land use planning, urban construction, and natural resource exploitation. We address the method for forming fuzzy land covers from TM images.

On TM images the pixel value is the reflectance of all spatial features per pixel. One pixel may contain different features. Therefore, the fuzziness of a land cover object is raised by both thematic and spatial resolution. Since the fuzziness in object definition and object extent cannot be differentiated from the value itself, the result of classification contains fuzziness in both thematic and spatial aspects of land cover objects.

6.3.2 Method for forming fuzzy land cover objects

Following the general procedure discussed in Section 6.2, a method for forming fuzzy land cover objects from TM images is proposed. The procedure is illustrated in Figure 6.3.

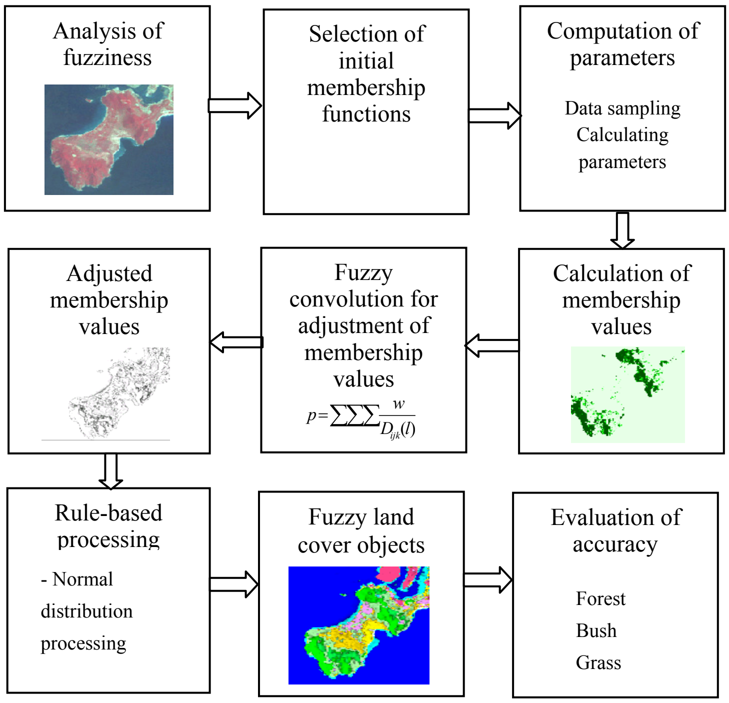


Figure 6.3 Procedure for forming fuzzy land cover objects from TM images

The method consists of seven steps: analyzing fuzziness in land covers, selecting appropriate initial membership functions, computing parameters of membership functions, fuzzy convolution for adjusting the membership values according to the land cover texture, rule-based processing for finalizing membership values, representing fuzzy land cover objects, and testing accuracy.

6.4 Test area

Sanya city is selected for testing the proposed method. The city is located on Hainan Island, in the south of China. The TM image was obtained on 18 April 1990. The study area ranges from E108.97257° N18.160917° to E109.550161° N18.5904583°, covering the city and rural areas, with 1960*2350 pixels on the TM image (Figure 6.4). The image in Figure 6.4 shows about 220*200 pixels.

Eleven land cover types will be classified from the 7-band TM image: *forest, bush, shrub and grassland, waste land, bare land, water body, beach, built-up area, rural area, paddy field, dry land*. The definition of these land cover types is as follows:

(1) Forest: in a pixel most trees are greater than 6 m, with the canopy generally covering over 80% of the pixel;
(2) Bush: in a pixel most trees are between 2 and 6 m, with the canopy generally covering over 50% to 80% of the pixel;
(3) Shrub and grassland: in a pixel there are some trees normally less than 2 m, with the canopy generally covering between 50 and 80% of the pixel;
(4) Waste land: in a pixel there are some trees less than 1 m, with the canopy generally covering between 10 and 50% of the pixel;
(5) Bare land: the canopy covers less than 10% of the pixel;
(6) Water body: a pixel covered by water;
(7) Beach: a pixel covered by wet sands and some water;
(8) Paddy field: a pixel covered by paddy fields;
(9) Dry land: a pixel covered by dry land;
(10) Built-up area: most of the pixel covered by buildings, roads or other construction material;
(11) Rural area: a pixel generally covered by building, with some trees, dry land, paddy fields or other land cover types.

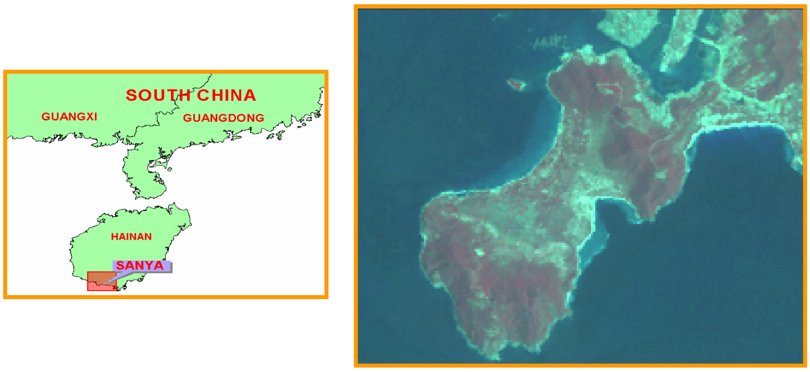


Figure 6.4 Location and TM image of test area

6.5 Generating fuzzy land cover objects

6.5.1 Selecting membership function

Several methods are available for deriving membership functions, such as the C-means clustering approach (Bezdek et al. 1984, Chi and Yan 1995), adaptive vector quantization (Dickerson and Kosko 1993), the self-organizing map approach (Chi *et al.* 1995), the fuzzy supervised classification approach (Wang 1990, Mannan *et al* 1998), and neural network approach (Sun and Jang 1993). The first three are clustering methods that generate the cluster center and variance. They are useful when the center should be calculated from all the data. The fuzzy supervised classification approach is mainly used to derive crisp classes. In this application, the conventional supervised classification is selected, since we assume that the membership function of each fuzzy land cover type basically follows the Gaussian distribution. It should be pointed out that neural networks (Gopal *et al.* 1999) and other classification approaches are also applicable.

6.5.2 Deriving initial membership values

The conventional maximum likelihood classifier is selected to calculate the initial membership values. The process includes data sampling, computing parameters for each membership function and calculating the membership values for each land cover type.

The maximum likelihood is a classic probability-based classifier and can be found in many remote sensing textbooks (Richards 2000). We adopt the following formula to calculate the weighted distance of pixel values belonging to a certain land cover type:

$$D_c = \ln(a_c) - 0.5\ln(|Cov(c)|) - 0.5(X - M_c)^T (Cov(c))^{-1}(X - M_c) \qquad (1)$$

where:

D_c is the weighted distance (the distance of a pixel belonging to class *c*)
c = a particular class
X = the measurement vector of the candidate pixel
M_c = the mean vector of the sample of class *c*
a_c = percent possibility that any candidate pixel is a member of class *c,* (defaults to 1.0, or is entered from *a priori* knowledge)
Cov(*c*) = the covariance matrix of the pixels in the sample of class *c*
|*Cov*(*c*) |= determinant of *Cov (c)*
$Cov(c)^{-1}$= inverse of *Cov c*
T = transposition function

The membership value of a pixel belonging to class *c* can be calculated by the following formula:

$$MV_c = \frac{MV_c}{\sum_{i=1}^{m} MV_i} \qquad (2)$$

where

$$MV_i = e^{D_i - \min(D_k)}, k = 1,2,...,m \qquad (3)$$

and *m* is the number of land cover types.

To compute the parameters, every land cover type is trained by supervised sampling. The maximum likelihood classifier classifies the land cover types when the pixel values of each type follow a Gaussian distribution. In some cases, a certain class has to be sampled by sub-classes such that each of them follows the Gaussian distribution. After classification, they can be merged together.

In the Sanya application, the paddy fields are split into two sub-classes, paddy field with water and paddy field with a lot of canopy of rice leaves, since there is a big difference between the pixel values of these sub-classes. Therefore in the classification, initially 12 types are sampled and classified.

The maximum likelihood classification adopts sample data to calculate the parameters of the membership function. It is well known that the sample data tremendously affect the classification results. Although several sampling methods are used to check the correctness of the classification, it should be pointed out that the final result is obtained by sampling data in small polygons conventionally. The classification is done using ERDAS Imagine.

In classification, the prior possibilities of all land cover types are assigned to value 1. We assume that each pixel may contain a maximum of four different land cover types. There are four layers of membership values for each pixel. The largest membership values are stored in the first layer, which shows the maximal membership value belonging to a class. The second layer stores the second largest membership values. The first layer of the classification result is partly shown in Figure 6.5. It also represents the crisp result of classification.

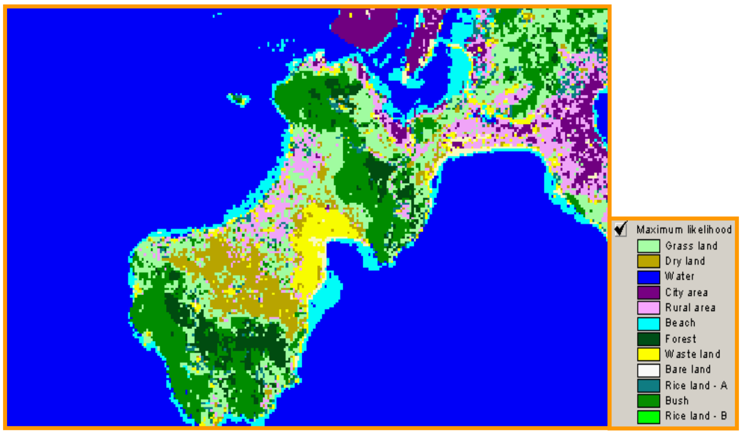


Figure 6.5 The class at the first layer after classification

6.5.3 Fuzzy convolution

After the classification, the weighted distances of each pixel belonging to every class are calculated. After checking the results, several errors can be observed. For example, some dry land has been classified as rural land; some land cover objects are too small. This is because the maximum likelihood classifier classifies images pixel by pixel. In order to derive a better result, the fuzzy convolution is applied to adjust the weighted distances of the pixel belonging to all classes, because it considers the membership values of neighboring pixels.

The basic formula of fuzzy convolution is:

$$T(c) = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{l=1}^{m}\frac{w_{ij}}{D_{ijl}(c)} \tag{4}$$

where:

$T(c)$ = the distance after fuzzy convolution to class c
w_{ij} = the weight of pixel (i,j)
c = a particular land cover class
m= number of land cover classes
$D_{ijl}(c)$ = the weighted distance of pixel (i,j) belonging to class c at layer l.
x,y = the number of neighborhood pixels

After the fuzzy convolution, the membership value of a pixel belonging to class c can be revised from formula (3) to the following:

$$MV_i = e^{\frac{\sum_{i=1}^{x}\sum_{j=1}^{y} w_{ij}}{T(c)} - \frac{\sum_{i=1}^{x}\sum_{j=1}^{y} w_{ij}}{\min(T(k))}}, k = 1,2,\ldots,m \tag{5}$$

In practice, a 3*3 matrix is adopted to adjust the membership values of land cover types. The matrix is:

$$\begin{bmatrix} 0.5 & 0.646 & 0.5 \\ 0.646 & 1.000 & 0.646 \\ 0.5 & 0.646 & 0.5 \end{bmatrix}$$

The sum of weight $\sum_{i=1}^{x}\sum_{j=1}^{y} w_{ij}$ = 5.584. The results of the fuzzy convolution are shown in Figure 6.6. After fuzzy convolution, the membership values are more continuous for the same class. In the Sanya application, the class “rice with a good canopy” and the class “rice with a normal canopy” are then merged together. The class “paddy field” is derived simply by adding the two membership values belonging to the two sub-classes for each pixel.

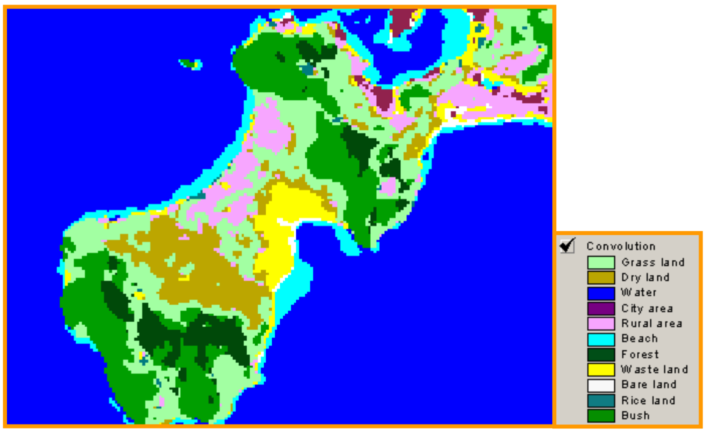


Figure 6.6 The first layer after fuzzy convolution

6.5.4 Additional adjustment of membership values

After these steps, fuzzy land cover objects are draft generated as shown in Figure 6.6. However, the above result is still not accurate enough. Figure 6.7 shows the extent of bush. In Figure 6.7 there are some pixels (marked in pink) where the membership values for bush cannot reflect the real situations. In general three problems can be detected:

(1) On some pixels the membership values for bush are very small. These pixels definitely belong to other classes, for example, to forest.

(2) There are also some pixels whose membership values for bush are very large but less than 1. These pixels actually must belong to bush.

(3) Some pixels definitely belong to a single class; however, the membership values for that class are far less than 1. For example, a pixel that must be dry land may just have a not very large membership value for grassland. However, the membership value is only 0.6 for dry land.

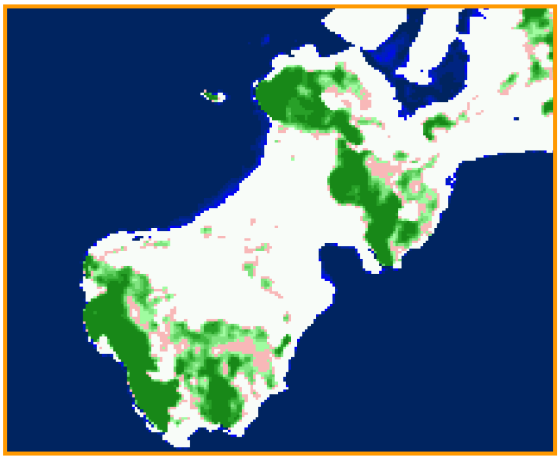


Figure 6.7 Spatial extent of land cover objects *bush*

The first two problems are caused by the maximum likelihood classifier. When this method is adopted, it is actually assumed that the membership function of a fuzzy land cover follows the Gaussian distribution absolutely and can be calculated by the trained data. In the general situation, this assumption is correct since most pixels follow the Gaussian distribution. However, this does not hold in extreme situations. Take a simple example. There is a pixel whose pixel value is 255 in each band; then the membership values for all land covers will be 10% when 10 classes are classified (Figure 6.8). However, it does not mean the membership values are 10% in practice, which explains why we assume that the membership values follow the Gaussian distribution just basically. The large and small values should be refined.

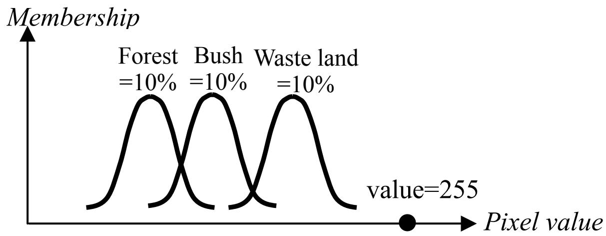


Figure 6.8 The pixel value (255) has the possibilities of 10% for all types

In order to solve the first two problems, we define thresholds to cut off small membership values and enlarge the big membership values according to the variance of the distribution. In the Gaussian distribution, if the membership value for a certain class is equal to 1, then the pixel covers all characteristics of that class. If the membership value of a pixel for a class is greater than 0.84, then the pixel falls within the interval of 1.5 variance (1.5σ) (Figure 6.9). We can cut off the two tails according to Figure 6.9. That is, if the membership value is not greater than 0.16, then it is set to 0. It means that, after classification, if the pixel covers only characteristics of a certain class to the extent 16%, then the membership value is changed to 0. On the other hand, if a pixel covers the characteristics to the extent 84% (=100%-16%), then the membership value is assigned to 1, showing that it definitely belongs to that class.

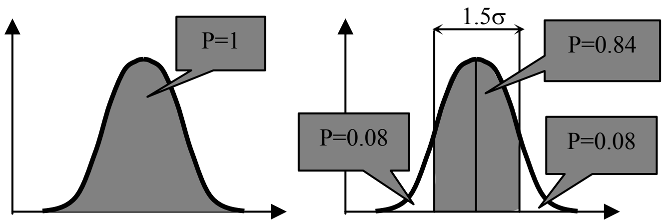


Figure 6.9 Possibility distribution of a fuzzy land cover object

Part of the result after the adjustment is shown in Figure 6.10.

The membership values between 0.16 and 0.84 are then normalized by a linear transformation:

$$MV_c^{'} = MV_c / 0.68 - 0.16 / 0.68 \qquad (6)$$

where MV_c is the original membership value and MV'_c is the new membership value. Parameters 1/0.68 and - 0.16/0.68 are calculated according to the assumptions: if MV = 0.84, then MV' = 1, and if MV = 0.16 then MV' = 0.

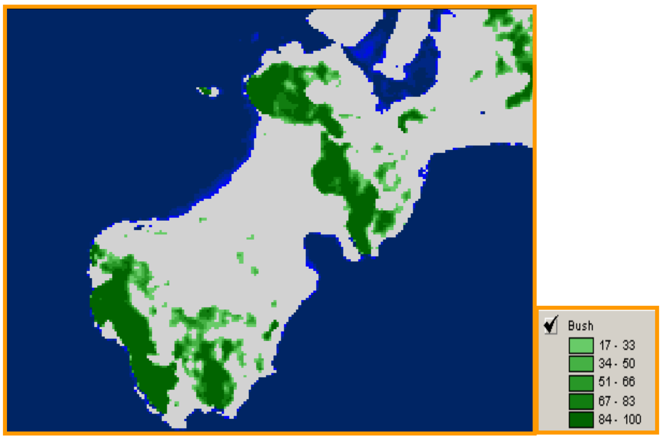


Figure 6.10 Cutting off and enlarging the membership values

6.5.5 Rule-based processing

In order to solve the third problem, more consideration should be given to definitions of land cover types. The membership values calculated from the above procedures are derived from the pixel value, where a Gaussian distribution is assumed for each class. However, when a certain land cover type is clearly defined, then there is no fuzziness in this class. Therefore, the fuzziness of a pixel belonging to this class is only caused by the spatial resolution. For example, a pixel has a membership value for dry land and a membership value for grassland. If the dry land is clearly defined, then the fuzziness is caused by the pixel size. That is, in one pixel, there is part dry land and part grassland. The maximum likelihood classifier actually calculates the membership values of a pixel for different classes.

In reality, some crops grow on dry land, which causes the pixel values to be similar between dry land and grassland or waste land. If we take the value of dry land and the value of grassland (for example) as the membership values (this means the pixel is composed of dry land and grassland), then a lot of errors can be detected through the comparison with the reality, since many of these pixels are actually either dry land or grassland. Because of the similarity of the pixel values between grassland and dry land, even if 300 clusters are classified by an unsupervised clustering, these two classes can still not be differentiated. Therefore, it is better to neglect the mixed pixels composed of grassland and dry land, since the dry land usually covers larger areas and in reality it seldom mixes with grassland. According to this assumption, it is reasonable to enlarge the dominant membership values. This strategy is useful when a lot of pixels that actually belong to one class have the values of two classes and it is difficult to differentiate between the membership values.

In practice, we assume that the definition of dry land and paddy fields are crisp, and neglect the mixed pixels between these two classes and other classes. Two rules can then be stated:

(1) If a pixel has membership values belonging to dry land and grassland, then it belongs to the class whose membership value is the maximum of the membership values, and the membership values are added up;

(2) If a pixel has membership values belonging to paddy field and beach, then it belongs to the class whose membership value is the maximum of the membership values, and the membership values are added up.

The final results are shown in Figure 6.11 and Figure 6.12.

6.5.6 Representing fuzzy land cover objects

After the above procedures, the fuzzy land cover objects are formed. The next step is to determine the boundary and the core of each object. The simplest way is to take the pixels whose membership values are greater than 0 as the extent of that type, and take the area of whose membership values are equal to 1 as the core and its boundary, since the core is an open set that does not contain its boundary. The difference between the core and the extent forms the boundary. The fuzzy land cover objects can be visualized in the following two ways.

Represent the land cover objects by different classes in different layers. On each layer, the change of colors represents the membership values. Figure 6.11 shows the membership values belonging to grassland. By this method, the distribution of any class can be visualized clearly, however, the overall classes on a pixel cannot be shown in one layer. This is useful for visualizing the distribution of a single class.

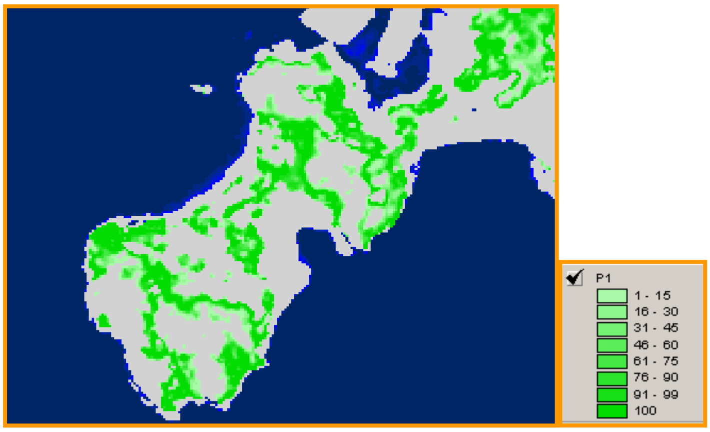


Figure 6.11 Representation of fuzzy land cover object *grassland*

Show the dominant classes in the first layer, using brightness to represent the membership values for that class (Figure 6.12). The second layer shows the second class that the pixel belongs to, also using the brightness to represent the membership values for that class. When there is no second class on the second layer (in the case of crisp objects), then the pixel will contain no data. The third layer can be done in the same way. In fact the layer is an extension of the visualization used in showing a crisp classification result, with the difference being that the membership value is shown on each pixel. The method is better for understanding the overall situation of spatial objects. However, the secondary classes cannot be shown in the same layer.

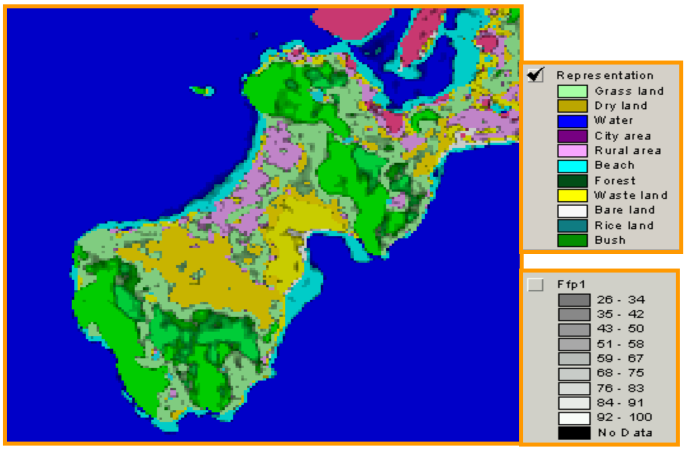


Figure 6.12 Representation of dominant land cover objects

6.6 Evaluation of accuracy

Compared with the methods for evaluating the accuracy of crisp objects, the theory of accuracy testing of fuzzy objects is still not mature. We evaluate the accuracy of fuzzy land covers by using air-photos, which are also verified by fieldwork. One hundred and sixty-two (162) sample points were selected from the area that is difficult to classify. The truth value of fuzzy land cover objects is estimated based on expert knowledge. At each point, membership values belonging to two land cover classes are simulated. Table 6.1 shows the accuracy results.

Several comparisons have been made based on test data. In Table 6.1, the first row compares results between membership values of the dominant class of the classification result with expert estimation at each point. One hundred and forty-three (143) points are classified correctly, *i.e.*, 88.2% of 162 points. This also represents the accuracy of the crisp classification. The average of the differences is 10 between the membership values, which is calculated based on the following formula:

$$\varepsilon_{MV_{C_1}} = \sum | MV_{c_1} - MV_{T_1} | / n$$

where $n = 143$.

It denotes that, if the pixel is correctly classified at the first layer, then the accuracy of the membership value is 90%.

Table 6.1 Evaluation of accuracy of fuzzy land cover objects

	Num ber	Percentage	Average of difference between membership values	Standard deviation between membership values
C1 = T1	143	88.2%	10	14
C1 = T2	3	1.8%	77	6
C2 = T1	1	0.6%	64	0
C2 = T2	121	74.7%	2	6
C1=T1 or C2=T1	144	88.8%	10	14
C1=T2 or C2=T2	124	76.5%	2	6
C1=T1 or C1=T2	147	90.7%	11	14
C2=T1 or C2=T2	122	75.3%	3	6
C1=T1 or C2=T1 or C1=T2 or C2=T2	160	98.8%		
C1=T1 and C2=T2	108	66.7%		
T1=Cover type 1 from air-photo (ground truth), C1= Cover type 1 from TM image, T2=Cover type 2 from air-photo (ground truth), C2=Cover type 2 from TM image				

The second row shows the comparison between the dominant class after classification and the secondary class of the ground truth. It means that the dominant class should be the secondary class of the true value. The average of the differences is 77 between two membership values.

The third row shows the comparison between the secondary class and the dominant class of the true value. It means that the secondary class should be the dominant class of the ground truth. The average of the differences is 64 between two membership values.

The fourth row gives the comparison between the secondary class and the true values. It shows that 121 points are correctly classified in the secondary layer, *i.e.*, 74.7%. The mean difference is 2, which denotes that the accuracy of membership values is 98% in the correct classes.

The fifth row shows the comparison between the classes in both the dominant and secondary layer and the dominant class of the true value. It means that, if any classified land cover type is equal to the true class, then the pixel is correctly represented. This comparison will add to the number of the pixels that are correctly classified, but enlarge the average difference in membership values.

Rows six to eight have a meaning similar to that of the fifth row. The ninth row shows the percentage that any one of the classes being equal to any of the true classes.

The tenth row denotes the percentage when both the dominant class and the secondary class are correctly classified. It indicates the accuracy of classification in both the dominant and secondary layers. The total overall accuracy is 66.7%.

From the above statistics, it can be seen that if the land cover class is correctly classified, then the differences are very small between classified results and the true classes. That is, the classification accuracy is the key that affects the accuracy of membership values of each land cover object.

6.7 Conclusions and discussions

6.7.1 Conclusions

This chapter discusses a general procedure for forming fuzzy spatial objects. Three steps are involved in this procedure. Based on this procedure, a method is provided to form fuzzy land cover objects. The procedures are further explained by classifying a TM image into fuzzy land cover types. While this chapter focuses on the method and its procedures, it also proposes some ways of representing fuzzy spatial objects, and discusses some aspects of evaluating the accuracy for fuzzy spatial objects.

In this chapter a composite method is proposed for form fuzzy land cover objects. Seven steps are involved: designing the initial membership functions for land cover types, generating initial parameters by supervised classification, membership value adjustment by fuzzy convolution and membership distribution, determining membership values by rule-based processing, representing of fuzzy land cover objects, and evaluating the accuracy of these objects. It can be seen that the key issue is how to derive the membership values of fuzzy land cover objects. According to the results of the test area and the accuracy evaluation, it is shown that the proposed method is suitable for forming fuzzy land cover objects.

6.7.2 Discussions

The chapter proposes a method of forming fuzzy land cover objects. The method can be revised in several aspects.

(1) The limitation on minimum membership values can be put at the classification stage. That is, small membership values can be filtered out through setting some confidence value before classification.
(2) Different weighted values can be adopted in the fuzzy convolution for different applications.
(3) The fuzzy convolution can also be done after filtering out the small membership values.
(4) The fuzzy spatial objects are formed by using fuzzy set theory; they can also be tuned by using a neural network approach.

Less consideration is paid to different classification methods. Actually, many classification methods are available, such as the knowledge-based classifier and neural network classification, which can also generate the membership values for fuzzy spatial objects. It is necessary to discuss them to compare the accuracy of land cover classes and membership values.

The accuracy of different land cover types is not presented in the chapter. Actually, how to investigate the accuracy of the fuzzy objects, and how to represent fuzzy spatial objects are two research aspects in modeling fuzzy spatial objects. Although some efforts have been made, more overall and systematical research is still needed.

Chapter Seven

Querying Fuzzy Spatial Objects

7.1 Introduction

An important difference between spatial data and conventional data lies in the spatial coherence between spatial data. The spatial coherence can be reflected in many aspects such as length, area, distance and angle. Topological relations between spatial data are one of these fundamental characteristics. Querying spatial objects through different topological relations is one of the basic tasks of GIS. In order to support fuzzy spatial objects in GIS, besides theoretic research on definitions of fuzzy spatial types, topological relations and the practical generation of fuzzy spatial objects, we should be able to query fuzzy objects in different ways, and especially query fuzzy spatial objects based on different kinds of topological relations.

In Chapters 4 and 5, the topological relations were formalized between simple fuzzy regions, simple fuzzy lines and fuzzy points. For example, between two simple fuzzy regions there are 44 topological relations when the empty/non-empty topological invariants are adopted. One hundred and fifty-two (152) topological relations between two simple fuzzy regions can be identified if the 4*4-intersection matrix is adopted.

In practice, it is necessary to query fuzzy spatial objects according to the topological relations between them. Since the topological relations between fuzzy spatial objects are all identified, the simplest method is to generate each operator according to these topological relations between two fuzzy objects. For example, we can define 44 or 152 query operators for the topological query.

However, as this will produce a large number of topological operators, the query will not be practical. In conventional GIS, a common method for querying spatial objects through topological relations is to summarize these topological relations in such a way that the query is easily understandable, yet it can meet most requirements of general applications.

Schneider (2001) proposed a method for querying fuzzy objects based on a set of crisp topological relations that are defined between two crisp objects created by α-cut from fuzzy objects, and whose membership values are derived from the membership values of fuzzy objects. In his method, the membership values are derived from each pair of different α-level crisp objects. This method is able to query fuzzy spatial objects. Strictly speaking, this query is not fuzzy topological (refer to Section 7.3.1). It is also difficult to query spatial objects with a specified relation with other objects. For example, it is very difficult to query spatial objects that have relation (6) in Appendix 1 with another fuzzy region.

In order to query fuzzy objects based on different topological relations, different methods are necessary. In this Chapter four methods are proposed for querying spatial objects. The operations address the querying of fuzzy regions, which can easily be extended to the querying of fuzzy lines and points. A combinatorial method and a fuzzy method are proposed based on the qualitative topological relations. The Schneider method is updated into a so-called crisp-relation-set-based fuzzy query and furthermore, the fuzzy-relation-set-based fuzzy query method is proposed for retrieving fuzzy objects. The four query methods provide relatively complete solutions to retrieving fuzzy objects for GIS applications.

The chapter is structured in the following way. Section 7.2 reviews some topological properties of fuzzy regions. Section 7.3 introduces some notions related to topological relations, including qualitative fuzzy topological relations and topological relation sets. Section 7.4 designs six operators from conventional GIS applications. Section 7.5 intensively discusses four query methods for retrieving fuzzy regions based on qualitative relations as well as on topological relation sets. Section 7.6 is the design of query interfaces and the query implementation. Section 7.7 analyzes the similarities and differences of different query methods. Conclusions and discussions are summarized at the end.

7.2 Topological properties of a fuzzy region

In Chapter 5 a fuzzy region was formally defined. It is quoted here for the next discussion. A fuzzy region is a fuzzy complex composed of fuzzy 2-cells of finite fuzzy complex such that the union of its cells meets the following conditions in $\widetilde{R}^2$:

(1) It is a non-empty proper double-connected closed set;
(2) The interior is a double-connected regular open set;
(3) The support is equal to the support of the closure of the interior;
(4) The core, the internal boundary and the outer are respectively a collection of subsets such that these subsets are mutually disjoint and every subset is double-connected.

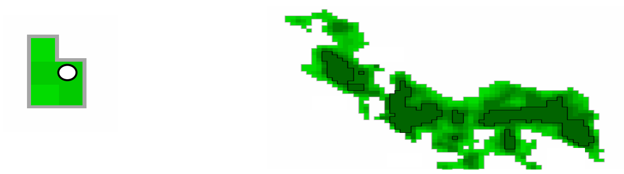


Figure 7.1 Two fuzzy objects *bush*

Figure 7.1 illustrates two fuzzy regions (*bush*) that are generated in Chapter 6.

Some of topological properties of a fuzzy region have been explained in the previous chapters. The properties that will be used in this chapter are summarized as follows (Let A be a fuzzy region in $\tilde{R}^2$.):

(1) The core $A^{\oplus}$: the core is the interior of the crisp subset of fuzzy region A in $\tilde{R}^2$. It is similar to the interior of a crisp region in R^2;
(2) The boundary of the core $\partial(A^{\oplus})$: it is the boundary of core of A. It is similar to the boundary of a crisp region in R^2;
(3) The boundary ∂A: it is the difference between A and the core of A;
(4) The frontier $\partial^e A$: it is the subset of a fuzzy region A for which the closure of the interior of A is greater than the interior of A;
(5) The internal boundary $\partial^i A$: it is equal to the interior of the boundary of A (refer Proposition 4.24(3));
(6) The external frontier $\partial^{ex} A$: it is the subset of A for which the fuzzy region is greater than the union of the closure of the core of A and the interior of A;
(7) Crisp region $\text{supp}(A)$: it is the support of A;
(8) α-level region A_α: it is a crisp set of A whose memberships are greater than and equal to α;
(9) Fuzzy α-level region $A_{\tilde{\alpha}}$: it is a fuzzy set of A whose membership values are the same as the fuzzy region if these values are equal to or greater than α, otherwise 0. We call it *fuzzy α-level region* (refer to fuzzy α-level set in Section 2.3.4).

The crisp region, the α-level region, and the fuzzy α-level region are topological properties in $\tilde{R}^2$, since they are all closed in $\tilde{R}^2$. These properties are illustrated in Figure 7.2.

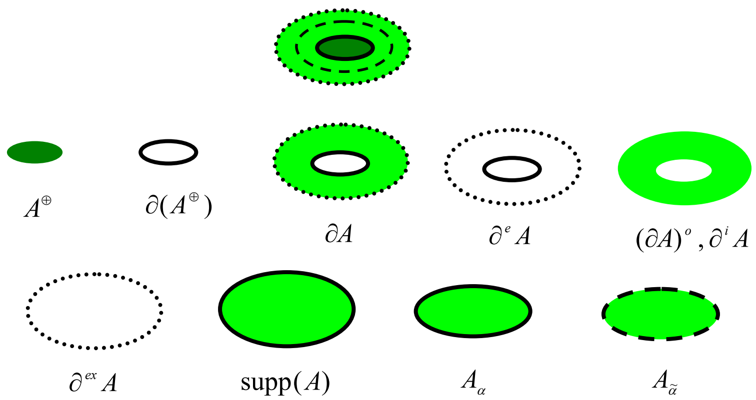


Figure 7.2 Topological properties of a fuzzy region

7.3 Qualitative topological relation and topological relation set

7.3.1 Qualitative topological relation

A *qualitative (fuzzy topological) relation* is the relation that is fuzzy topological and the membership value of the relation is 1. According to this definition, between two simple regions, the 44 topological relations that are identified based on the 3*3-intersection matrix, and the 152 topological relations that are identified based on the 4*4-intersection matrix by using empty/non-empty topological invariants are qualitative relations, since the result is a collection of empty and non-empty values. The 77 topological relations between two real simple fuzzy regions are also qualitative relations.

7.3.2 Topological relation set

We define a notion *topological relation set*. The first type of topological relation set is defined based on crisp relations between two fuzzy objects. As we know, the crisp relations hold between two α-level (crisp) objects. Let $(\widetilde{X}, \delta)$ be an fts of *X*, define a mapping $F: \widetilde{X} \to X$ from fts $(\widetilde{X}, \delta)$ to cts $(X, [\delta])$ such that $F = \{x : x = x_{\alpha}, if\ \alpha > 0; x = 0\ if\ \alpha = 0\}$. Then for every subset $A \subseteq \widetilde{X}$, we get a set of α-level crisp sets $\{A_{\alpha} : \alpha > 0\}$ of *A* in cts $(X, [\delta])$. Let *A, B* be two fuzzy regions in $(\widetilde{X}, \delta)$. Through this mapping, we will get two collections of crisp regions A_{α} and B_{α}. Between each pair (A_{α} , B_{α}) there is a crisp topological relation. The collection of crisp topological relations between all pairs $\{(A_{\alpha}, B_{\alpha}) : \alpha > 0\}$ of each level is called a *crisp topological relation set*. Note *F* is not a fuzzy homeomorphic mapping. Therefore, the collection of topological relations between these crisp regions is not a fuzzy topological relation between two fuzzy regions in the fuzzy topological space. We just assume that the fuzzy topological relation *r* between two fuzzy regions *A* and *B* is depicted by a set of crisp topological relations r_{cr} between two α-level regions A_{α} and B_{α} of *A* and *B*: $r(A, B) = \{r_{cr}(A_{\alpha}, B_{\alpha}) : \alpha > 0\}$. This definition is slightly different from Schneider's method. He proposed that the crisp topological relations include every relation between crisp regions A_{α} and B_{β} $(\alpha, \beta \in [0,1])$. In our definition, the crisp topological relations are derived between A_{α} and B_{α}, where each pair of A_{α} and B_{α} is at the same level.

The second type of the topological relation set is defined based on a set of qualitative relations between two fuzzy regions. In the fts, we can derive a collection of fuzzy α-level (sub-)regions, *i.e.*, a fuzzy set *A* is the union of fuzzy subsets: $A = \{A_{\tilde{\alpha}} : 0 \le \alpha \le 1\}$. Between two fuzzy (sub-)regions $\{A_{\tilde{\alpha}}, B_{\tilde{\alpha}}\}$, one of the qualitative relations holds when the 3*3- or 4*4-intersection matrix is applied. The collection of these qualitative relations consists of fuzzy topological relations. Note that every qualitative relation is fuzzy topological; the collection of qualitative relations between two fuzzy (sub-)regions can really represent the fuzzy topological relation between two fuzzy regions. We call this set *fuzzy topological relation set*.

The crisp topological relation set and the fuzzy topological relation set are illustrated in Figure 7.3. An α-level region A_α of fuzzy region *A* is marked in grey. An α-level region B_α is marked in green. $A_{\tilde{\alpha}}$ is a fuzzy α-level region of *A*. $B_{\tilde{\alpha}}$ is a fuzzy α-level region of *B*. The crisp topological relation set is the collection of crisp topological relations between A_α and B_α of every level. The fuzzy topological relation set is the collection of fuzzy qualitative topological relations between $A_{\tilde{\alpha}}$ and $B_{\tilde{\alpha}}$ of every level.

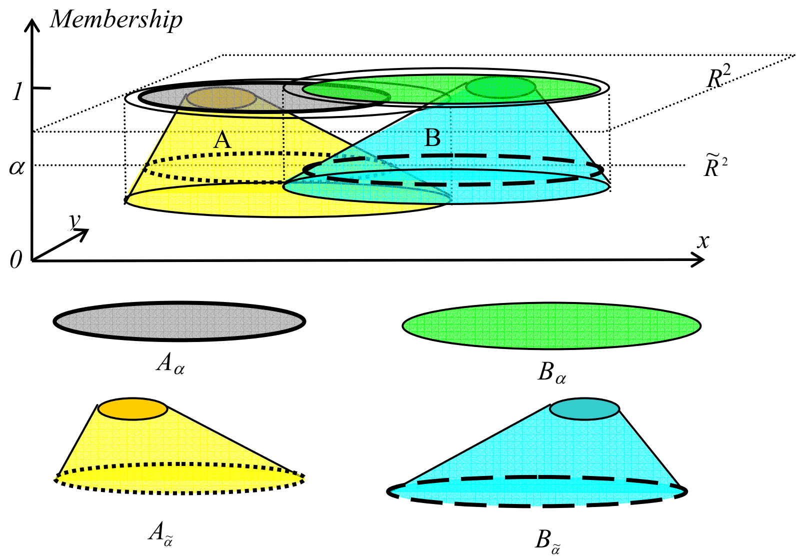


Figure 7.3 Topological relation set

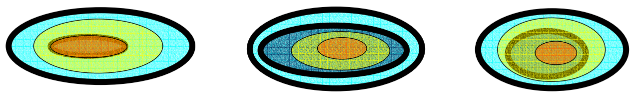
We will demonstrate the methods for querying fuzzy regions based on qualitative relations as well as two kinds of topological relation sets. The discussion on querying fuzzy lines is omitted, since the query operations are similar. Two kinds of query methods are proposed: query based on qualitative relations and query based on two types of relation sets.

7.4 Query operators

7.4.1 Design of query operators

The 152 or 44 relations between two simple fuzzy regions are all qualitative. These relations can also be identified between two fuzzy regions. Generating every operator for each relation is not practical because there are too many query operators. It is necessary to design appropriate query operators so that fuzzy spatial objects can be retrieved, and the query can be easily understood.

Therefore, the query operators should be defined. The conventional eight topological relations {*disjoint, contains, inside, covers, coveredby, equal, meet, overlap*} form a group of the best candidates to be considered as fuzzy query operators since they are mathematically defined and have already become the common-sense geography. However, they should be adjusted in fuzzy settings. When we look into the eight relations, it can be found that the relations *cover* and *contains* share the difference as to whether both boundaries are intersected or not (the other intersection conditions are the same). In the fuzzy situation, since the boundary of a fuzzy object may have the same dimension as the object itself, it is impossible to differentiate between these two relations according to the boundary-boundary intersection. For example, Figure 7.4 shows three settings between two simple fuzzy regions. Is it possible to tell the difference between setting *A, B* and *C*?



A B C

Figure 7.4 Three qualitative relations between two simple fuzzy regions

The 3*3-intersection matrices are expressed by $\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$ respectively.

If we look at the first two rows and the first two columns, the setting *B* is relation *contains* but *A* and *C* are *covers* in terms of the 4-intersection matrix in the cts. However, most people regard them as the same. To make the query operators easily understandable, the *contains* and *covers* relations are merged together. For the same reason, the relations *inside* and *coveredby* are not differentiated. Therefore, we define six *basic relations*: {*disjoint, contains, inside, equal, meet, overlap*} (in short*, basic relations*) to be query operators.

The definitions of the basic relations are generated according to the intuitional meaning of a topological relation. When a relation is mentioned, an intuitional meaning of this relation will be reflected in our minds. For example, when we consider the *meet* relation, we will think that the meaning of *meet* is that the boundaries intersect each other and the interiors are disjoint.

We can interpret the intuitional meaning of the basic relations between two fuzzy regions *A* and *B* as follows:

(1) *Disjoint: A and B are disjoint*;
(2) *Meet: The boundaries intersect each other, and the two cores are disjoint;*
(3) *Contains: A contains B;*
(4) *Overlap: A is not inside B and A does not contain B and vice versa;*
(5) *Inside: A is inside B;*
(6) *Equal: A is equal to B.*

These intuitional meanings can be transferred into the formal definitions for these basic relations. Mathematically they can be defined by the valuation sets $\{0,1\}$, $\{0\}$ and $\{1\}$. $\{0,1\}$ means that the value could be either 0 or 1.

(1) *A disjoint B:* $\begin{bmatrix} \{0\} & \{0\} & \{0,1\} \\ \{0\} & \{0\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix}$;

(2) *A equal B*: $\begin{bmatrix} \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{0\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix}$;

(3) *A contains B*: $\begin{bmatrix} \{0,1\} & \{0,1\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{0\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix}$;

(4) *A inside B*: $\begin{bmatrix} \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{0\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix}$;

(5) *A meet B*: $\begin{bmatrix} \{0\} & \{0,1\} & \{0,1\} \\ \{0,1\} & \{1\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0,1\} & \{0,1\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{0\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix}$;

(6) *A overlap B:* $\begin{bmatrix} \{0,1\} & \{0,1\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0,1\} & \{0,1\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{0\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix} - \begin{bmatrix} \{0\} & \{0,1\} & \{0,1\} \\ \{0,1\} & \{1\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix} -$
$\begin{bmatrix} \{0\} & \{0\} & \{0,1\} \\ \{0\} & \{0\} & \{0,1\} \\ \{0,1\} & \{0,1\} & \{1\} \end{bmatrix}$.

These definitions partly result from the crisp topological relations between two crisp regions. For example, the *contains* relation between two crisp regions *A* and *B* can be formalized by $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ in cts. The intersections between cores and boundaries are the unique values $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and the interior and the boundary of *B* do not intersect the exterior of *A*. In fuzzy settings, since the boundary of a fuzzy region may be two-dimensional, the intersections between boundaries and cores have more variances than the unique values in the cts. However, the interior and the boundary of *B* cannot intersect the exterior of *A* when *A* contains *B*. If we consider fuzzy regions *A* and *B* as a whole, then *A contains B* can be defined by the fact that the interior and the boundary of *B* do not intersect the exterior of *A*, and intersections between cores and boundaries are either 0 or 1. It is easy to show that each basic relation is a fuzzy topological relation since it is preserved under fuzzy homeomorphisms.

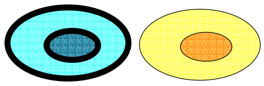
7.4.2 Grouping qualitative relations

Based on these definitions the qualitative relations can be grouped into the basic relations. The concept of grouping topological relations in crisp situation was formally proposed by Clementini *et al* (1993). We group the 44 relations between two simple fuzzy regions into the basic relations. These relations can also be detected between two fuzzy regions.

Appendix 1 lists these 44 qualitative relations between two simple fuzzy regions. Tables 7.1 to 7.6 show the result of grouping these qualitative relations in the six basic relations.

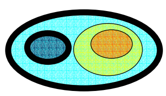
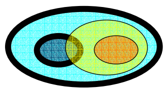
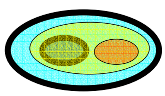
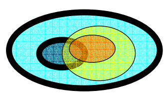
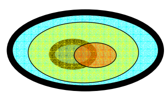
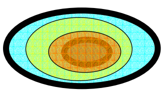
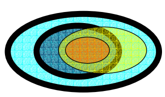
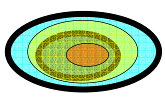
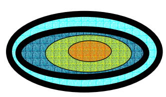
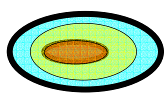
(1) *A disjoint B*: Only relation (1) of the 44 qualitative relations is detected to belong to this class based on the definition of the basic relation.

Table 7.1 A disjoint B

$(1)\begin{bmatrix}0&0&1\\0&0&1\\1&1&1\end{bmatrix}$	

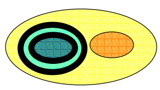
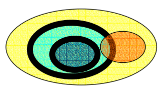
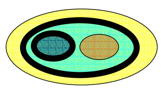
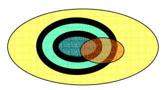
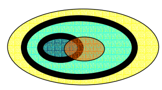
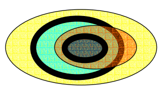
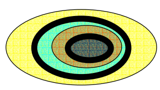
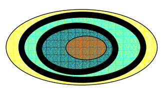
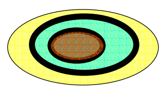
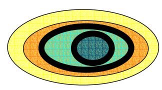
(2) *A contains B*: Ten relations belong to this class.

Table 7.2 A contains B

$(8)\begin{bmatrix}0&0&1\\1&1&1\\0&0&1\end{bmatrix}$ 	$(13)\begin{bmatrix}0&1&1\\1&1&1\\0&0&1\end{bmatrix}$ 	$(15)\begin{bmatrix}0&1&0\\1&1&1\\0&0&1\end{bmatrix}$ 	$(22)\begin{bmatrix}1&1&1\\1&1&1\\0&0&1\end{bmatrix}$ 	$(25)\begin{bmatrix}1&1&0\\1&1&1\\0&0&1\end{bmatrix}$ 
$(31)\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix}$ 	$(33)\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$ 	$(36)\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$ 	$(40)\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}$ 	$(43)\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$ 

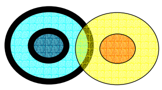
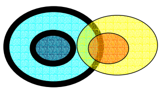
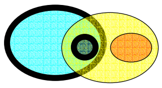
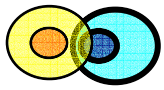
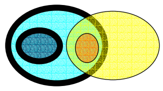
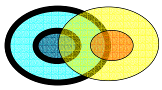
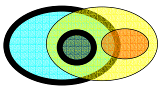
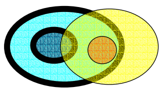
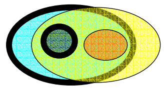
(3) *A inside B*: Ten relations are also detected.

Table 7.3 A inside B

$(5)\begin{bmatrix}0&1&0\\0&1&0\\1&1&1\end{bmatrix}$ 	$(11)\begin{bmatrix}0&1&0\\1&1&0\\1&1&1\end{bmatrix}$ 	$(16)\begin{bmatrix}0&1&0\\1&1&0\\0&1&1\end{bmatrix}$ 	$(20)\begin{bmatrix}1&1&0\\1&1&0\\1&1&1\end{bmatrix}$ 	$(24)\begin{bmatrix}1&1&0\\1&1&0\\0&1&1\end{bmatrix}$ 
$(27)\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$ 	$(30)\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$ 	$(37)\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix}$ 	$(39)\begin{bmatrix}1&0&0\\1&0&0\\1&1&1\end{bmatrix}$ 	$(42)\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$ 

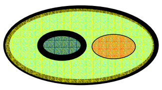
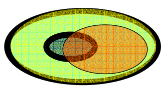
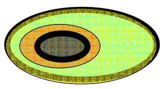
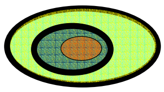
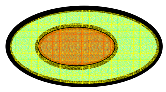
(4) *A meet B*: There are nine relations.

Table 7.4 A meet B

(2) $\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$ 	(3) $\begin{bmatrix}0&1&1\\0&1&1\\1&1&1\end{bmatrix}$ 	(4) $\begin{bmatrix}0&1&0\\0&1&1\\1&1&1\end{bmatrix}$ 	(6) $\begin{bmatrix}0&0&1\\1&1&1\\1&1&1\end{bmatrix}$ 	(7) $\begin{bmatrix}0&0&1\\1&1&1\\0&1&1\end{bmatrix}$ 
(9) $\begin{bmatrix}0&1&1\\1&1&1\\1&1&1\end{bmatrix}$ 	(10) $\begin{bmatrix}0&1&0\\1&1&1\\1&1&1\end{bmatrix}$ 	(12) $\begin{bmatrix}0&1&1\\1&1&1\\0&1&1\end{bmatrix}$ 	(14) $\begin{bmatrix}0&1&0\\1&1&1\\0&1&1\end{bmatrix}$ 	

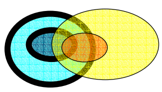
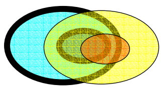
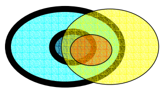
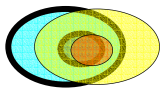
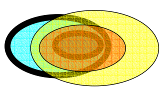
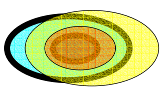
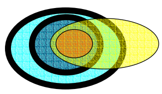
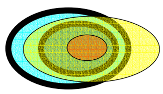
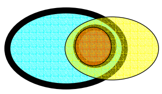
(5) *A equal B*: There are five relations.

Table 7.5 A equal B

(17) $\begin{bmatrix}0&1&0\\1&1&0\\0&0&1\end{bmatrix}$ 	(26) $\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$ 	(32) $\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$ 	(38) $\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ 	(41) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ 

(6) *A overlap B*: There are nine relations.

Table 7.6 A overlap B

(18) $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ 	(19) $\begin{bmatrix}1&1&0\\1&1&1\\1&1&1\end{bmatrix}$ 	(21) $\begin{bmatrix}1&1&1\\1&1&1\\0&1&1\end{bmatrix}$ 	(23) $\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}$ 	
(28) $\begin{bmatrix}1&0&0\\1&1&1\\1&1&1\end{bmatrix}$ 	(29) $\begin{bmatrix}1&0&0\\1&1&1\\0&1&1\end{bmatrix}$ 	(34) $\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$ 	(35) $\begin{bmatrix}1&1&0\\0&1&1\\0&1&1\end{bmatrix}$ 	(44) $\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$ 

7.5 Methods for querying fuzzy regions

7.5.1 Combinatorial query method

If the six basic relations are adopted as query operators to query fuzzy spatial objects, the fuzzy spatial objects that have a basic relation with others will be retrieved. For example, if the operator is *overlap*, a query "retrieve the fuzzy regions that overlap with fuzzy region *A*" will get all the fuzzy regions that have the topological relation *overlap* with *A*, which includes relations (18), (19), (21), (23), (28), (29), (34), (35) and (44) listed in Table 7.6. There is no possibility of retrieving fuzzy regions based on a specified relation, for instance (18).

7.5.1.1 Absolute sub-relation

In order to be able to retrieve a fuzzy region based on a specified qualitative relation, we introduce the notion *(absolute) sub-relation*. As we know, the boundary and the core of a (simple) fuzzy region are two-dimensional if it is not a crisp region; and the fuzzy region itself is also two-dimensional. They are all topological invariants. We can derive the basic relations between two invariants of two fuzzy regions. Let *A* be a fuzzy region. There are three two-dimensional topological invariants: *A* itself, the core of *A*, and the boundary of *A*. For two fuzzy regions *A* and *B*, there are nine pairs between these components: *A* and *B*, *A* and $B^{\oplus}$, *A* and ∂B, ∂A and *B*, ∂A and ∂B, ∂A and $B^{\oplus}$, ∂A and $B^{\oplus}$, $A^{\oplus}$ and ∂B, and $A^{\oplus}$ and ∂B.

Between fuzzy regions *A* and *B*, if we adopt the external frontier $\partial^{ex} A$ of *A*, the union ($\partial^{i} A \cup A^{\oplus-}$) of the interior of the boundary and the closure of the core of *A*, the external frontier $\partial^{ex} B$ of *B*, the union ($\partial^{i} B \cup B^{\oplus-}$) of the interior of the boundary and the closure of the core of *B* to form a 2*2-intersection matrix, then the eight relations {*disjoint, contains, inside, cover, coveredby, equal, meet, overlap*} can be identified. These can be grouped into six basic relations: {*disjoint, contains, inside, equal, meet, overlap*}.

If we adopt the frontiers ($\partial^{e} A$, $\partial^{e} B$) and the interior of the boundary ($\partial^{i} A, \partial^{i} B$) for ∂A and ∂B, then the same basic relations can be identified between *A* and ∂B, ∂A and *B*, ∂A and ∂B.

Between *A* and $B^{\oplus}$, seven relations can be identified except relation *meet*: {*disjoint, contains, inside, cover, coveredby, equal, overlap*}. These can be grouped into five basic relations*: {disjoint, contains, inside, equal, meet, overlap*}. The same basic relations exist between ∂A and $B^{\oplus}$, ∂A and $B^{\oplus}$, $A^{\oplus}$ and ∂B, $A^{\oplus}$ and ∂B.

A basic relation between these components of fuzzy region *A* and fuzzy region *B* is called a *(absolute) sub-relation* between *A* and *B*.

By using a certain combination of these sub-relations, the 44 or 152 relations between two simple fuzzy regions can be achieved. This method is called the *combinatorial* *query method.*

7.5.1.2 Query based on qualitative relations

When we apply a query based on the 44 relations, the combination of four sub-relations is enough: $A^{\oplus}$ and B, $A^{\oplus}$ and $B^{\oplus}$, A and $B^{\oplus}$, and A and B. For example, if we want to query fuzzy regions based on relation (22) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ of the 44 relations (all 44 relations are listed in Tables 7.1 to 7.6 of Section 7.4), it can be detected by three sub-relations: *A contains B*, $A^{\oplus}$ *overlap B*, and $A^{\oplus}$ *overlap* $B^{\oplus}$ (Figure 7.5).

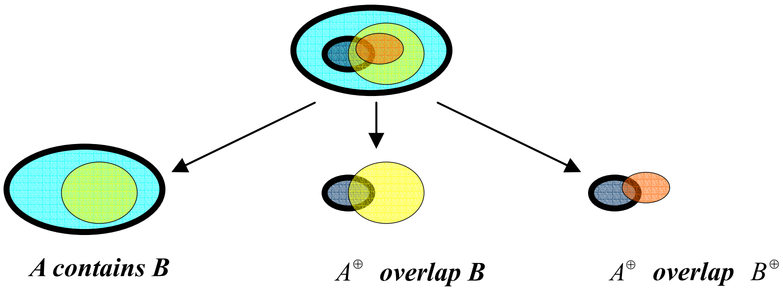


Figure 7.5 Three sub-relations of topological relation (22) between fuzzy simple regions *A* and *B*

To identify 152 relations, four more relations should be added: $A^{\oplus}$ and ∂B, ∂A and $B^{\oplus}$, and ∂A and ∂B.

The 152 or 44 relations between two simple fuzzy regions can be achieved by the above combinations of basic relations between the components. Between two fuzzy regions, more topological relations can be derived, yet these relations can be grouped into 152 or 44 relations. And they can be achieved by the above combinatorial method.

It is also possible to query other qualitative topological relations between two fuzzy regions, for example, the qualitative relation that is defined between a fuzzy α-level region *A* of a simple fuzzy region and another fuzzy region *B*. The qualitative relations between *A* and *B* can be formalized by the 3*3-intersection matrix or the 4*4-intersection matrix. For example, we can make the forest area whose membership values are greater than and equal to 0.2 a fuzzy 0.2-level region, and apply the 3*3-intersection or 4*4-intersection matrix to detect if this region has a kind of relation with another region.

In order to query fuzzy regions based on membership value comparisons within a qualitative topological relation, four operations can be generated: $\supseteq$, $\subseteq$, $\neq$, = as defined in Section 4.5.4. For example, we can ask whether the membership degrees of a fuzzy region are greater than those of another fuzzy region when they are overlapped spatially.

7.5.2 Qualitatively-based fuzzy query method

The above approach retrieves fuzzy regions directly based on the qualitative relations, which are accomplished through the combinations of sub-relations between the topological properties of fuzzy regions. The query operation is crisply implemented such that each sub-relation is crisply valued and the combination of these sub-relations is just a crisply valued set. This approach is useful for querying fuzzy objects based on the exact qualitative relations. However, since there are 44 and 152 relations, we might not know which relations we are really concerned with; on the other hand, we might get confused when we want to retrieve some fuzzy objects according to some combinations of sub-relations, since there are many combinations. Sometimes it is not very practical since this approach requires some ideas on notions of fuzzy spatial objects.

7.5.2.1 Relative sub-relations

Another way of querying fuzzy spatial objects is to retrieve them based on a basic relation with fuzziness, *i.e.*, a basic relation is a fuzzy relation. Let us investigate an example. There are two fuzzy simple regions in Figure 7.6. What is the relationship between them?

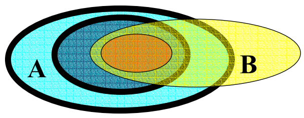


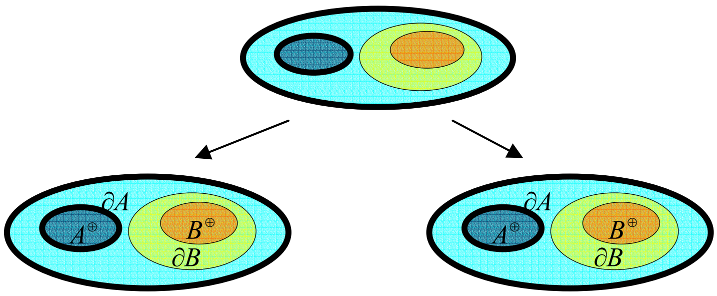
Figure 7.6 Two fuzzy regions

The 3*3-intersection matrix indicates that it is relation (34)$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. If we look at the first two rows and two columns $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, it basically belongs to the relation *contains* in terms of crisp topological relations. However, one might regard *A overlap B*. Apparently both opinions are reasonable since it does not equal any 9-intersection matrix in Section 3.3.1. Although there are 44 or 152 relations between two fuzzy simple regions, these 44 or 152 relations just express the crisp information, and the fuzziness of the relation is not interpreted.

In order to query fuzzy objects according to fuzziness in spatial objects, it is useful to introduce another query method. We still adopt the six basic relations as query operators. But each operator is considered to be a fuzzy relation with the membership values indicating the membership degree or values for the relation. For example, we can take the basic relation *contains* as a fuzzy relation that represents *contains* and whose membership value indicates the degree it belongs to *contains*. The membership degrees of these operators are generated based on qualitative relations between fuzzy regions. We design a qualitatively-based fuzzy query method for querying fuzzy regions. Then the question is how to assign the membership degrees for these basic relations from qualitative relations.

A fuzzy approach is adopted to realize these operators. It detects so-called relative sub-relations between some components of fuzzy regions; they are then classified into the basic relations; and finally the membership degrees are assigned for each relation. This query method is called the *qualitatively-based fuzzy query method*.

As we know, the qualitative relations are formed based on the intersections of their topological parts and it is also possible to derive the topological relations (which are called sub-relations) to be the basic relations between these parts. We introduce another notion: the relative sub-relation. *A relative sub-relation* is a relation that is identified between two two-dimensional topological components of two fuzzy regions, with relativity to the regions themselves. A relative sub-relation is formed in the following way: the definitions of all these components are unchanged and the relation is formed according to the definition of the basic relations. In the previous section, we saw that there are three two-dimensional topological parts to a simple fuzzy region *A*: *A* itself, the core and the boundary of *A*. Between two fuzzy regions *A* and *B*, there are nine kinds of relative sub-relations. We can identify the relative sub-relations between these parts, for example, the relative sub-relations between $A^{\oplus}$ and *B*, and $A^{\oplus}$ and $B^{\oplus}$ for relation (8). Since $A^{\oplus}$ and ∂B do not intersect, and both boundaries intersect, $A^{\oplus}$ and *B* *meet* each other (because $A^{\oplus}$ does not intersect ∂B, and both boundaries intersect). $A^{\oplus}$ and $B^{\oplus}$ *meet* also each other (because both cores do not intersect and both boundaries intersect) (Figure 7.7).



$A^{\oplus}$ ***meets B*** $\qquad$ $A^{\oplus}$ ***meets*** $B^{\oplus}$

Figure 7.7 Two relative sub-relations of topological relation (8) between fuzzy simple regions *A* and *B*

Apparently the relative sub-relation between *A* and *B* is a basic relation. In other words, we have one kind of basic relation. In order to make a fuzzy query to retrieve fuzzy regions based on the 44 qualitative relations, four kinds of relative sub-relations *A* and *B*, $A^{\oplus}$ and *B*, $A^{\oplus}$ and $B^{\oplus}$, *A* and $B^{\oplus}$, are necessary and sufficient. The necessity is because some relations need all these relative sub-relations to be differentiated such as relations (18), (19), (28) and (29). The sufficiency is because the other relative sub-relations are implicitly contained in these relative sub-relations.

In the above way, four relative sub-relations can be grouped into the basic relations, respectively. The process of the classification is shown as follows and the results are summarized in Table 7.7. The relation (1) is omitted since all relative sub-relations are *disjoint*.

Table 7.7 Basic relations for 44 qualitative relations in terms of relative sub-relations

Qualitative relation No.	$A^{\oplus}$ and B	$A^{\oplus}$ and $B^{\oplus}$	Qualitative relation No.	A and $B^{\oplus}$	$A^{\oplus}$ and $B^{\oplus}$	Qualitative relation No.	$A^{\oplus}$ and $B^{\oplus}$
(8)	Meet	Meet	(5)	Meet	Meet	(17)	Meet
(13)	Overlap	Meet	(11)	Overlap	Meet	(26)	Overlap
(15)	Inside	Meet	(16)	Contains	Meet	(32)	Inside
(22)	Overlap	Overlap	(20)	Overlap	Overlap	(38)	Contains
(25)	Inside	Overlap	(24)	Contains	Overlap	(41)	Equal
(31)	Inside	Inside	(27)	Overlap	Inside		
(33)	Overlap	Contains	(30)	Contains	Inside		
(36)	Inside	Contains	(37)	Contains	Contains		
(40)	Contains	Contains	(39)	Inside	Inside		
(43)	Inside	Equal	(42)	Contains	Equal		
Basic relation between A and B is *contains*; Basic relation between A and $B^{\oplus}$ is *contains*.			Basic relation between A and B is *inside*; Basic relation between $A^{\oplus}$ and B is *inside*.			Basic relation between A and B is *equal*; Basic relation between A and $B^{\oplus}$ is *contains;* Basic relation between $A^{\oplus}$ and B is inside.	

Qualitative relation No.	A and $B^{\oplus}$	$A^{\oplus}$ and B	Qualitative relation No.	A and $B^{\oplus}$	$A^{\oplus}$ and B	$A^{\oplus}$ and $B^{\oplus}$
(2)	Meet	Meet	(18)	Overlap	Overlap	Overlap
(3)	Overlap	Meet	(19)	Overlap	Inside	Overlap
(4)	Meet	Inside	(21)	Contains	Overlap	Overlap
(6)	Meet	Overlap	(23)	Contains	Contains	Overlap
(7)	Contains	Meet	(28)	Overlap	Inside	Inside
(9)	Overlap	Overlap	(29)	Contains	Inside	Inside
(10)	Overlap	Inside	(34)	Contains	Overlap	Contains
(12)	Contains	Overlap	(35)	Contains	Inside	Contains
(14)	Contains	Contains	(44)	Contains	Inside	Equal
Basic relation between A and B is *meet*; Basic relation between $A^{\oplus}$ and $B^{\oplus}$ is *meet*.			Basic relation between A and B is *overlap*.			

By using this approach, each of the 44 qualitative relations is classified into the four types of relative sub-relation.

7.5.2.2 Membership values of relations between simple fuzzy regions

Between two simple fuzzy regions, the membership degrees for each basic relation are then assigned as follows. Since in total four relative sub-relations between A, $A^{\oplus}$, B and $B^{\oplus}$ are adopted, the membership degree of 0.25 can be assigned to each relative sub-relation, so that the sum of membership values for each qualitative relation is 1. For example, for the qualitative relations (8), (13), (15), (22), (25), (31), (33), (36), (40) and (43), the basic relations between A and B, and A and $B^{\oplus}$ are 0.25. If the basic relations of relative sub-relations are the same, the membership degrees can be added together. Table 7.8 lists the membership degrees of basic relations for 44 qualitative relations.

Table 7.8 Membership degrees of basic relations for 44 qualitative relations

Relation	C	I	M	E	O	Relation	C	I	M	E	O
(8)	0.5		0.5			(2)			1		
(13)	0.5		0.25		0.25	(3)			0.75		0.25
(15)	0.5	0.25	0.25			(4)		0.25	0.75		
(22)	0.5				0.5	(6)			0.75		0.25
(25)	0.5	0.25			0.25	(7)	0.25		0.75		
(31)	0.5	0.5				(9)			0.5		0.25
(33)	0.75				0.25	(10)		0.25	0.5		0.25
(36)	0.75	0.25				(12)	0.25		0.5		0.25
(40)	1					(14)	0.5		0.5		
(43)	0.5	0.25		0.25		(18)					1
						(19)		0.25			0.75
(5)		0.5	0.5			(21)	0.25				0.75
(11)		0.5	0.25		0.25	(23)	0.5				0.5
(16)	0.25	0.5	0.25			(28)		0.5			0.5
(20)		0.5			0.5	(29)	0.25	0.5			0.25
(24)	0.25	0.5			0.25	(34)	0.5				0.5
(27)		0.75			0.25	(35)	0.5	0.25			0.25
(30)	0.25	0.75				(44)	0.25	0.25		0.25	0.25
(37)	0.5	0.5				(17)	0.25	0.25	0.25	0.25	
(39)		1				(26)	0.25	0.25		0.25	0.25
(42)	0.25	0.5		0.25		(32)	0.25	0.5		0.25	
						(38)	0.5	0.25		0.5	
						(41)				1	
C=*contains*, I=*inside*, M=*meet*, E=*Equal*, O=*overlap*											

From Table 7.8, it can be perceived that for some qualitative relations, membership degrees of basic relations are equal. For example, in Relation (14), the membership values for *contains* and *meet* are 0.5 and some relations (Relation 40, 39, 2, 18, 41) are crisp. This matches our intuition. Actually, relations (40), (39), (2), (18) and (41) are crisply *contains*, *inside*, *meet*, *overlap* and *equal,* respectively. Therefore, this method is reasonable. The membership values can also be partly checked by using topological distance between two simple regions (Egenhofer and Franzosa 1994).

7.5.2.3 Membership values of relation between fuzzy regions

This method is also applicable to querying fuzzy regions based on the qualitative relations. More qualitative relations exist between two fuzzy regions. They can still be grouped into the six basic relations since they are crisply defined. The membership values of every basic relation can be calculated by the following steps. Suppose region *A* has *m* $A^{\oplus}$ and region *B* has *n* $B^{\oplus}$ ($m, n \in N$):

(1) Since *A* and *B* are unique, we cannot average the membership value for each basic relation according to the total number of combinations of relative sub-relations between these components. However, since there are four types of relative sub-relations between *A*, $A^{\oplus}$, *B* and $B^{\oplus}$: $r(A,B)$, $r(A^{\oplus},B)$, $r(A,B^{\oplus})$, and $r(A^{\oplus},B^{\oplus})$ where r ={*disjoint, contains, inside, equal, meet, overlap*}, we can assign the membership value for each basic relation by $\pi_r = 0.25$, *i.e.*, $\pi_r(A,B) = 0.25$, $\pi_r(A^{\oplus},B) = 0.25$, $\pi_r(A,B^{\oplus}) = 0.25$ and $\pi_r(A^{\oplus},B^{\oplus}) = 0.25$.

(2) Between *A* and *B*, since they are unique, the membership value of the basic relation between *A* and *B* is 0.25. Between $A^{\oplus}$ and *B*, there are *m* relative sub-relations. The membership value of each basic relation is calculated by:

$$\pi_r(A_i^{\oplus},B) = \frac{0.25}{m} (1 \le i \le m) \tag{1}$$

The membership values will be added up when the basic relations are the same, *i.e.*, if $r(A_i^{\oplus},B) = r(A_j^{\oplus},B), 1 \le i,j \le m, i \ne j$, then the membership value for *r* is $\pi_r(A_i^{\oplus},B) + \pi_r(A_j^{\oplus},B) = \frac{0.25}{m} + \frac{0.25}{m} = \frac{0.5}{m}$.

(3) The basic relations between *A* and $B^{\oplus}$ can be calculated according to (2).

(4) Between $A^{\oplus}$ and $B^{\oplus}$, there are maximally $m*n$ kinds of relative sub-relations. If there is a core $A_i^{\oplus}$ $(1 \le i \le m)$ that is *equal* to (or is *inside*) $B_s^{\oplus}$ $(1 \le s \le n)$, then $A_i^{\oplus}$ does not intersect all $B_k^{\oplus}$ $(k = 1,2,...,n, k \ne s)$. We will not count all these *meet* relations. The number of relative sub-relations will be reduced from $m*n$ to $m*n-n+1$. Similarly, if there is a core $A_t^{\oplus}$ $1 \le t \le m$ that *contains* $B_j^{\oplus}$ $(1 \le j \le n)$, then $B_j^{\oplus}$ *meets* all $A_k^{\oplus}$ $(k = 1,2,...,m, k \ne j)$. We also remove them for the total number of relative sub-relations. In general, if there are some cores $A_i^{\oplus}$ that *equal* or are *inside* $B_s^{\oplus}$, and $A_t^{\oplus}$ *contains* some cores $B_j^{\oplus}$ $(1 \le i,t \le m)$ $(1 \le s,j \le n)$, the total number will be:

$$sum = m*n - (n-1)*\mathrm{count}(r^x(A_i^{\oplus},B_s^{\oplus})) - (m-1)*\mathrm{count}(r^y(A_t^{\oplus},B_j^{\oplus})) \tag{2}$$

where $\text{count}(\cdot)$ is used to count the number of relative sub-relations, $r^x(A_i^\oplus, B_s^\oplus)$ is *equal* or *inside*, and $r^y(A_i^\oplus, B_j^\oplus)$ is *contains*. The membership between $A_i^\oplus$ and $B_j^\oplus$ is then $\pi_r(A_i^\oplus, B_j^\oplus) = \frac{0.25}{sum}$. The membership values can also be added up if the basic relations are the same.

For example, there are two regions *A* and *B* illustrated in Figure 7.8. Then the membership value that *A contains B* is the sum of membership values that *A contains B*, and *A contains* $B^\oplus$: 0.25+0.25 = 0.5. Since two cores $A^\oplus$ do not intersect *B*, the *meet* relation membership value is 0.125 + 0.125 = 0.25. Similarly, since two cores $A^\oplus$ do not intersect $B^\oplus$, the *meet* relation membership value is also 0.25. Therefore the total membership value that *A meet B* is 0.25 + 0.25 = 0.5.

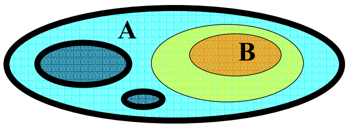


Figure 7.8 Calculations of membership values for relative sub-relations between two fuzzy regions *A* and *B*

7.5.3 Crisp-relation-set-based fuzzy query method

The above two methods are designed to query fuzzy objects based on qualitative relations. In Section 7.4 two types of topological relation sets are introduced. We now generate the query operations and their membership values based on these two types of relation sets.

According to the definition of the crisp relation set, eight topological relations can be adopted as the fuzzy relation. We propose the *crisp-relation-set-based fuzzy query method,* in which the operators are derived from eight relations, and membership values are calculated based on membership values of two fuzzy regions. Since the relation names are imported from the crisp topological relations, it is feasible to calculate the membership values based on the crisping properties and membership values of fuzzy regions. We adopt the concept of basic probability assignment for calculation (Schneider 2001). Let *F* be a fuzzy region. A basic probability assignment $m(F_{\alpha_i})$ can be associated with each α-level (crisp) region F_{α_i} and can be interpreted as the probability that F_{α_i} is the "true" representative of fuzzy region *F*, and is defined as

$$m(F_{\alpha_i}) = \alpha_i - \alpha_{i+1} \quad (3)$$

for $1 \le i \le n$ where $n \in N$ with $\alpha_1 = 1$ and $\alpha_{n+1} = 0$. That is, *m* is built from the differences of successive α_i's. It is easy to see that the sum $\sum_{i=1}^{n} m(F_{\alpha_i}) = 1$.

Let $\pi_f(A, B)$ be the value that represents a binary property between two fuzzy simple regions *A* and *B*. Supposing that each α-level region of *A* and *B* is a simple crisp region, then this membership value can be determined as the sum of weighted operations by

$$\pi_f(A,B)=\sum_{i=1}^{n} m\cdot\pi_{cr}(A_{\alpha_i},B_{\alpha_i}) \quad (4)$$

where $\pi_{cr}(A_{\alpha_i},B_{\alpha_j})$ yields the value of the corresponding property for two crisp α-level regions. The value $\pi_{cr}(A_{\alpha_i},B_{\alpha_j})$ is either 1 or 0, which is determined by the relations between two crisp α-level regions. $\pi_{cr}(A_{\alpha_i},B_{\alpha_j})\in T_{cr}$. T_{cr} = {*disjoint, contains, inside, cover, coveredby, equal, meet, overlap*}. For example, if two 0.2-level regions overlap, then $\pi_{cr}(A_{0.2},B_{0.2})=1$ for *overlap*, and 0 for all other relations.

Take an example, there are two fuzzy regions *A* and *B* illustrated in Figure 7.9.

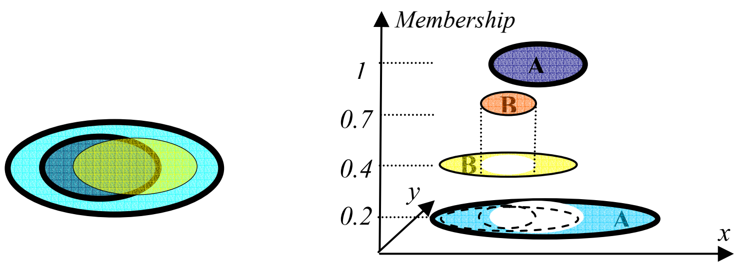


Figure 7.9 Calculation of membership values according to crisp-relation-set-based fuzzy query method

The membership values are:

$\pi_{overlap}(A,B)=$ (0.4-0.2) + (0.7-0.4) + (1-0.7) = 0.8

$\pi_{contains}(A,B)=$ 0.2

And the membership values of all other relations are zero.

In practice, the *cover* and *coveredby* relations are seldom derived between two fuzzy regions. They are merged into the *contains* and *inside* relations respectively.

Formula (4) is derived based on the condition that each α-level region is a simple crisp region. If a α-level region has several simple crisp regions, we can change formula (4) into (5) to derive membership values for each basic relation by:

$$\pi_f^{type\,I}(A,B)=\sum_{i=1}^{n}\sum_{j=1}^{k_i}\frac{m\cdot\pi_{cr}(A_{\alpha_{i,j}},B_{\alpha_{i,j}})}{k_i} \quad (5)$$

where k_i is the number of crisp relations between two α-level regions. And each k_i can be calculated according to formula (2). This is also the formula for calculating membership values of basic relations between two fuzzy regions.

7.5.4 Fuzzy-relation-set-based fuzzy query method

The above method derives the relations by crisping fuzzy objects at each level. It is also possible to query based on the fuzzy relation set introduced in Section 7.3. The fuzzy relation set is a set of qualitative fuzzy topological relations between two fuzzy α-level regions. However, since the number of qualitative relations is too big, it is necessary to reduce the relation numbers.

We propose a fuzzy-relation-set-based method to query fuzzy regions in the following steps. Firstly we detect the qualitative relations between two fuzzy α-levels; then we group qualitative relations into certain operators according to some rules; and then we generate the membership values for each of the operators according to the fuzziness of fuzzy regions. In Section 7.4 the six basic relations are defined and they are fuzzy topological; therefore they can be adopted as the operator names. According to their definitions, all qualitative relations can be grouped into one of these basic relations. We propose a *fuzzy-relation-set-based fuzzy query method,* in which each operator is a basic relation that is derived from the qualitative relation between two fuzzy α-level regions, and membership values of the basic relation are derived from the membership values of two fuzzy regions.

Supposing there are two fuzzy regions *A* and *B*, we can cut fuzzy regions *A* and *B* at each α-level into a set of two fuzzy regions $A_{\tilde{\alpha}}$ and $B_{\tilde{\alpha}}$ ($0 \le \alpha \le 1$). Between each pair of $A_{\tilde{\alpha}}$ and $B_{\tilde{\alpha}}$, a qualitative relation can be identified, which can be grouped into the six basic relations. Define a basic probability assignment *m* between $F_{\tilde{\alpha}_i}$ for a fuzzy region *F* by

$$m(F_{\tilde{\alpha}_i}) = \alpha_i - \alpha_{i+1} \tag{6}$$

for $1 \le i \le n$ where $n \in N$ with $\alpha_1 = 1$ and $\alpha_{n+1} = 0$. *m* represents the differences of successive $\alpha_i's$. The membership value of a basic relation can be calculated by

$$\pi_f^{type\ II}(A,B) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{m \cdot \pi(A_{\tilde{\alpha}_{i,j}}, B_{\tilde{\alpha}_{i,j}})}{k_i} \tag{7}$$

where $\pi(A_{\tilde{\alpha}_{i,j}}, B_{\tilde{\alpha}_{i,j}})$ represents a basic relation between $A_{\tilde{\alpha}_{i,j}}$ and $B_{\tilde{\alpha}_{i,j}}$ and is assigned by either 1 or 0. For example, if two 0.2-level fuzzy regions *overlap*, then $\pi(A_{0.2}, B_{0.2}) = 1$ for the basic relation *overlap*, 0 for the other basic relations. k_i is the number of basic relations between two fuzzy α-level regions. Each k_i can be calculated according to formula (2).

7.6 Querying fuzzy objects

7.6.1 Unary topological operators

Unary topological operators are used to derive the topological properties of a fuzzy object. According to discussions in Section 7.3, the basic unary operations can be realized based on the basic properties of a fuzzy region. They are also applicable for a fuzzy line and a fuzzy point (see Table 7.9).

Table 7.9 Basic properties of a fuzzy region, a fuzzy line and a fuzzy point

Operator	Operand	Result	Explanation
Core	(fuzzy) region /line	crisp region/line	
Boundary of the core	(fuzzy) region /line	crisp line/point	
Boundary	(fuzzy) region line	(fuzzy) region/line	The membership values can be attached to each point of the boundary
Frontier	(fuzzy) region/line	(fuzzy) line/point	The membership values can be attached to each point of the frontier
Internal boundary	(fuzzy) region/line	(fuzzy) region/line	The membership values can be attached to each point of the internal boundary
External frontier	(fuzzy) region/line	(fuzzy) line/point	The membership values can be attached to each point of the external frontier
Crisp object	(fuzzy) region/line/point	crisp region/line/point	
α-level object	(fuzzy) region/line/point	crisp region/line/point	
Fuzzy α-level object	(fuzzy) region/line	(fuzzy) region/line	The membership values can be attached to each point of the object

7.6.2 Binary topological operators

Binary topological operators can be derived according to four query methods. The operators between two fuzzy regions are summarized in Table 7.10. These are also applicable to querying fuzzy lines and fuzzy points.

7.6.3 Interface design

The interface is designed to fulfill the above query operations to retrieve fuzzy objects. Three different kinds of interfaces can be considered: property representation, property calculation and query interfaces. Property representation is used to show the properties of fuzzy objects. Different types of objects require different variables to be represented. The property calculation interface is designed to calculate different properties of fuzzy objects. The query interfaces will help to query fuzzy objects according to some conditions. Five query interfaces are designed: (1) basic query interface for querying according to the properties of one object, (2) combinatorial query interface, (3) qualitatively-based fuzzy query interface, (4) crisp-relation-set-based query interface, and (5) fuzzy-relation-set-based query interfaces.

Table 7.10 Binary topological operations

Group		Operator	Operand	Operand	Result	Explanation
Topological	Combinatorial query method	contains, inside, meet, equal, overlap, disjoint	fuzzy region /line /point	fuzzy region / line/ point	Boolean	
	Qualitatively-based fuzzy query method	contains, inside, meet, equal, overlap	fuzzy region /line /point	fuzzy region /line /point	Real	The result is a relation with membership value for each basic relation.
		disjoint	fuzzy region /line /point	fuzzy region /line /point	Boolean	
	Crisp/fuzzy-relation-set-based fuzzy query methods	contains, inside, meet, equal, overlap, disjoint	fuzzy region /line /point	fuzzy region /line /point	Real	The result is a relation with membership value for each basic relation.

7.6.4 Query implementation

Four query interfaces are illustrated in Figure 7.10. The basic query interface is omitted since it is the same as the query dialog of ArcView. The unary operations are not illustrated. In Figure 7.10(A), the properties are the core, the boundary and the object itself. The query operators in all interfaces are six basic relations. The membership degrees in the qualitatively-based query interface are assigned by a language variable: *somewhat* for 0.25, *basic* for 0.5, *mostly* for 0.75, and *absolute* for 1.

Different methods can be adopted for different purposes. If we want to retrieve fuzzy land cover objects (let's say *bush*) that have relation (44) with a certain dry land object, the combinatorial methods can be adopted. We may specify that the *selected dry land overlaps with bush, and overlaps with the core of bush, and the core of dry land overlaps with bush, and the core of dry land overlaps with the core of bush.* If we just want to retrieve fuzzy bush objects that basically overlap the selected dry land in general, we can adopt the second method to retrieve them. We can specify that the *selected dry land overlaps with bush* by membership degree *basic*. If we want to retrieve the fuzzy bush objects that overlap the selected dry land above a certain level, we can adopt either the third method or the fourth method for the query. We can specify that *the selected dry land overlaps with bush* and input a *membership value*.

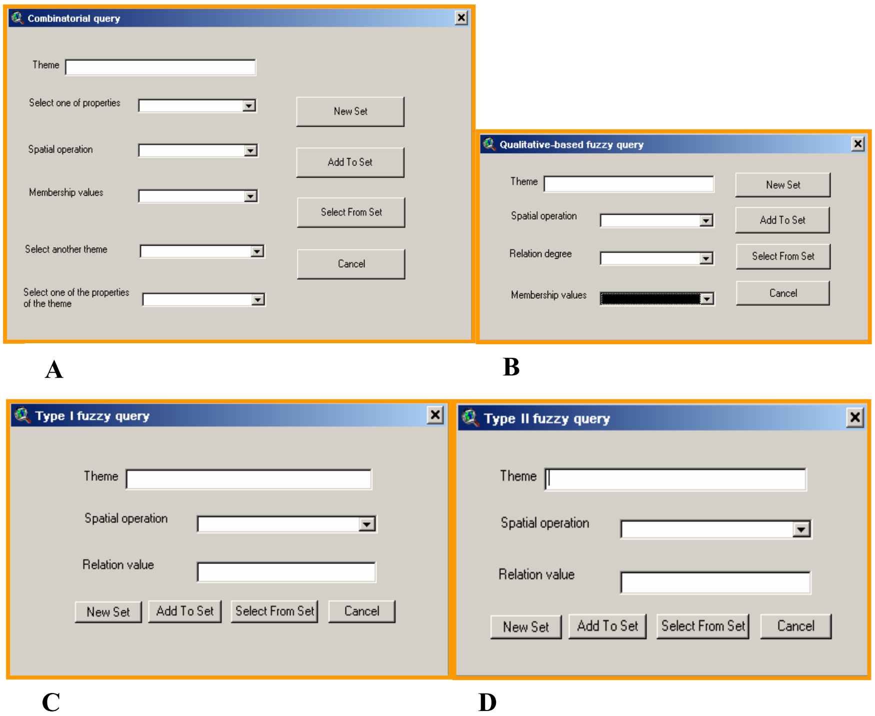


Figure 7.10 Four query interfaces
A. Combinatorial query method; B. Qualitatively-based fuzzy query method; C. Crisp-relation-set-based query method; D. Fuzzy-relation-set-based query method

7.7 Comparison on query methods

In Section 7.5, four methods are proposed for querying fuzzy regions based on the topological relations between two fuzzy regions. Different methods will retrieve different fuzzy objects. The combinatorial method can be adopted when we want to make a query based on the exact qualitative relation. The second method derives fuzzy objects according to qualitative relations and fuzziness in general situations. This method is suitable for querying when we just want to get the objects in general. For example, if we want to retrieve the objects that are basically inside another object, then we can set the operation to *contains* and the membership degree to *basic*. The crisp/fuzzy-relation-set-based methods can retrieve fuzzy objects more precisely.

The first approach works by a precise specification. The others query objects by using a fuzzy relation. In these three methods, the basic relations are the same but the meanings and membership values may have some differences.

The relation names in the qualitative-based query method and the fuzzy-relation-set-based query method have the same meaning. However, their membership values are derived in different ways. In the qualitatively-based query, membership degree is qualitatively assigned. In the fuzzy-relation-set-based query, the membership degree is precisely calculated based on the membership values of two fuzzy objects. These membership values cannot be compared since they have different premises. Actually the fuzzy-relation-set-based query method is a precise version of the qualitatively-based query method, in the sense that the membership values are calculated based on the membership values of fuzzy regions.

The relation meaning in the crisp-relation-set-based query is different from that in the qualitatively-based and fuzzy-relation-set-based query methods. The crisp-relation-set-based query is derived based on the crisp topological relations between α-level regions of fuzzy regions whereas the other two queries are based on the fuzzy topological relations. Therefore, there are some differences. For example, between the two fuzzy regions *A* and *B* illustrated in Figure 7.11, the membership values will be not zero for the two basic relations, *A overlap B* and *A disjoint B*, where other membership values are zero, in terms of the crisp-relation-set-based relation. However, the membership values will not be zero for the basic relations *A meet B*, and *A disjoint B* if the fuzzy-relation-set-based query method is applied. Both operators are applicable and practical.

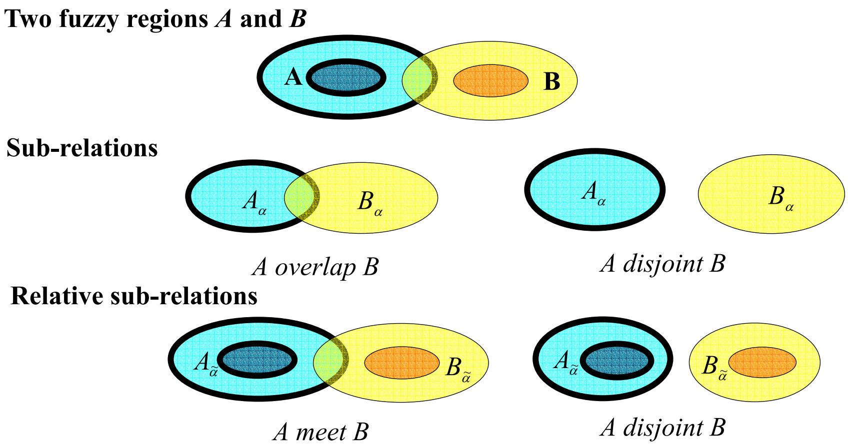


Figure 7.11 Comparisons between sub-relations and relative sub-relations

7.8 Conclusions and discussions

This chapter focuses on the topological operations and their implementation for querying fuzzy spatial objects. These operations are all applicable to retrieving these objects in GIS models.

The topological operations between two fuzzy objects are more complicated than those between two crisp objects. Between two fuzzy objects there is a qualitative fuzzy topological relation, which can be formed based on the intersection matrices in which the crisp values are adopted. The crisp topological relation set can also be adopted to describe the topological relation between two fuzzy objects. The fuzzy topological relations can also be represented by a set of qualitative fuzzy topological relations between the fuzzy α-level objects of two fuzzy objects.

In order to query fuzzy regions according to these topological relations and topological relation sets, the query operators should be defined first. Six basic relation names are adopted as the query operators for query operations, since these basic relations have their intuitional meaning in GIS and they originate from eight crisp topological relations between crisp regions.

Four querying methods are proposed, namely, the combinatorial method, the qualitatively-based fuzzy method, the crisp-relation-set-based method, and the fuzzy-relation-set based method for queries. They all adopt six basic relations as query operators. The first method is combinatorial, which is fulfilled by using different sub-relations between different topological properties such as cores and boundaries of two fuzzy regions. These operations are crisp.

The second method qualitatively derives the membership degree of basic relations from qualitative topological relations, according to the absolute sub-relations. The absolute sub-relations are derived between different topological parts of two fuzzy regions.

In the third method, the membership values of operators are calculated based on the crisp topological relation set between the α-level regions of two fuzzy regions. The eight crisp topological relations between the α-level regions are changed into six operations.

The fourth method is to query fuzzy objects based on the fuzzy topological relation set, whose members are basic relations that are derived by grouping qualitative relations, and the membership values are calculated according to the membership values of fuzzy regions in the basic relation.

The analysis starts from the relations between two simple fuzzy regions, then the formulae are extended for queries between two fuzzy regions. All these methods can be adopted to query fuzzy regions.

Different queries retrieve fuzzy objects in different ways. Different results will be achieved when different query methods are adopted. They are useful for retrieving fuzzy objects for different purposes.

Since these query methods have different theoretic backgrounds, the results of queries are different. Method selection needs to be tested more in practice, especially the choice between crisp- and fuzzy-relation-set-based methods. The data structure for accommodating fuzzy objects is not mentioned. The optimal query strategy and algorithms should be further researched.

Chapter Eight

Reasoning about Changes of Land Cover Objects

8.1 Introduction

In Chapters 6 and 7, two practical issues were investigated: the generation of fuzzy spatial objects and they query of these objects. This chapter will discuss the advantage of fuzzy spatial objects for understanding land cover changes.

Currently, spatial objects are represented by crisp forms in most applications. In some cases, the spatial objects are crisp, for example administrative boundaries. There are more cases where it is not necessary to represent spatial objects in fuzzy forms, since the applications just need to investigate the area size and so on. The crisp forms will simplify the representation of spatial objects so that processing these spatial objects will be relatively easy, and the result of analysis based on the crisp forms is sufficient for the applications.

There are also some applications that need fuzzy spatial objects. As we mentioned in Chapter 1, if we have to analyze the transitional changes of land cover objects, it is better to model land cover objects in fuzzy forms, otherwise the changes from one land cover to another will be represented in abrupt changes.

This chapter will discuss how to detect changes of land covers based on fuzzy settings. A two-step fuzzy reasoning method is proposed for calculating changes of land covers. The fuzzy land cover objects are generated from bi-temporal TM images. These data sets contain two kinds of errors: geometric error due to misregistration, and misclassification error. The second error is reflected by the membership values of fuzzy land cover objects. In order to minimize misregistration error, fuzzy polygons (fuzzy regions) are adopted instead of pixels in the first-step reasoning; in order to minimize classification error, the difference between spectral values is applied in the second-step reasoning.

The structure of this chapter is organized as follows. Section 8.2 generally reviews the change detection methods. Section 8.3 discusses the two-step reasoning methods for change detection. Section 8.4 shows fuzzy land covers generated based on the methods described in Chapter 6. Section 8.5 shows the results, and comparisons with the conventional method. Section 8.6 gives conclusions and discussions.

8.2 Review of change detection methods

Change detection using land use and land cover maps is the basis of much land cover dynamics research. Specifically, a wide variety of remote sensing methods have been developed for detecting land use and land cover change in bi-temporal categorical and multi-spectral imagery (Gong 1993, Metternicht 1999, Power 2001, Petit and Lambin 2001). Maybe the simplest method of change detection is to sum the differences between the spectral values of every band of the bi-temporal TM images. This method is available in almost all remote sensing processing software. The calculation result can precisely reflect the spectral change degree that is implicitly caused by the spatial object changes. However, because of the complexity of object reflection, the same object may reflect a different spectrum at different times, or different objects may reflect the same spectrum at different times. Therefore, in practice spectral difference is always taken to be a reference.

A more conventional method of change detection is to compare the differences based on the classified images; this is called *post-classification comparison*. It performs a *pixel-by-pixel overlay* of two thematic maps to generate a similarity map and associated statistics that indicate regions of disagreement of spatial objects. However, there are numerous examples in the literature of concerns about the limitations of the traditional methods. Power (2001) pointed out the following limitations:

"One problem with post classification comparison is that the accuracy and usefulness of the comparison results depend on the accuracy of the categorical classifications and geometric registration of the maps. A second, more important, limitation is that the traditional methods can only compare maps that contain Boolean categories. By nature, land use patterns are often inherently complex and can consist of an intricate intermixture of land use types. Boolean maps must frequently simplify or otherwise misrepresent land use patterns, so that the result of a post classification comparison may be imprecise. The accuracy of a comparison procedure based on a more reliable and robust approach could have a marked improvement in the ability to detect and model real world change.

A third problem with the traditional approaches is that, because they are based on a pixel-by-pixel comparison, they do not necessarily capture the qualitative similarities between two maps."

In order to avoid the above limitations, he pointed out:

"In contrast, the flexibility of a fuzzy representation of spatial data offers the potential for avoiding the problems of traditional comparison procedures. First of all, misregistration and locational inaccuracies can be accounted for by fuzzifying the

boundaries of the pixels or polygons of the input maps. Generally, the width of the fuzzy boundaries will correspond to the level of uncertainty in each of the land use maps. Using fuzzy implication algorithm, fuzzy polygons can be compared to determine the sections that are different due to error and those that are different because of actual land use disagreement (Edwards and Lowell 1996). Secondly, fuzzy set theory provides a method of dealing and comparing maps containing a complex mixture of spatial information. A fuzzy map is more appropriate for representing a complex land use type, such as vegetation coverage, because it enables the pixels or polygons to have multiple memberships in the land use classes. Furthermore, a fuzzy map comparison model can determine the agreement between fuzzy maps while handling the complexity of the land use classes rather than simply ignoring it. Therefore, the degrees and types of categorical differences between maps should be determined by a fuzzy post classification comparison."

According to his ideas, the fuzzy representation of spatial data is more suitable for change detection. In order to get a fuzzy representation of spatial land use objects, he proposed a hierarchical fuzzy pattern matching method to emulate human reasoning when comparing multiple maps. In his method, two land use maps are overlapped, and the fuzzification is done at the polygon level by subjectively defined membership functions.

As we pointed out in Chapter 1, land use land cover objects are fuzzy in nature. There is no clear boundary between one category of land cover objects and another. Therefore, it is better to represent land cover objects directly in fuzzy representation. We will propose a method in which the land cover objects are fuzzy and the change is detected based on fuzzy reasoning methods.

8.3 Methods for reasoning about land cover changes

A methodology for change detection is proposed based on fuzzy reasoning, and consists of the following steps: (1) generation of fuzzy land cover objects and derivation of differences of membership values; (2) reasoning about changes of categorical fuzzy land covers, based on fuzzy polygons (fuzzy regions); (3) reasoning about changes of land cover objects.

1. Generation of fuzzy land cover objects and derivation of differences in membership values

(1) Generation of fuzzy land cover objects

In Chapter 1 the methods for generating fuzzy spatial objects are categorized into two kinds: subjective and objective. In order to generate fuzzy land cover objects from TM images, an objective method is adopted, and used in Chapter 6. Basically the membership value is calculated for land cover objects at the pixel level by using maximum likelihood classification. That is, for every pixel, there might be several membership values, corresponding to different land cover types. For example, there is a pixel whose membership values are 0.7 for forest, and 0.3 for bush. This means we will not refuse a pixel with several characteristics. This is a more reasonable reflection of the real world, where there is no crisp boundary between two land cover objects. The boundary is a broad area from one category to another.

For simplicity, two membership values are allowed for each pixel. If the membership value is 1 for one category, then the membership value of the other will be 0. It means that the pixel definitely belongs to only one category. If the membership value is less than 1 for one category, then the membership value is the complement for the other category.

(2) Calculation of membership differences

After classification, the membership values of every spatial object are derived. For conventional crisp land cover objects, the difference between membership values is either 0 when the two land cover objects have the same category, or 1 when the categories of two land cover objects are different. If the classifications and the registration are all error-free, then the change can be detected directly by the difference between two land cover maps. In fuzzy settings, the situation is more complicated. A pixel may have more than one membership value. For example, a pixel has membership values 0.7 for forest and 0.3 for bush at time *A*; these change into 0.6 for grassland and 0.4 for waste land at time *B*. Several possibilities will arise, such as the difference between forest and grassland, and the difference between forest and waste land.

Since the changes are complicated, it is almost impossible to tell which category has changed into the other. Therefore, the differences between the membership values of different categories do not help the change detection. However, the differences can be compared if the membership values are of the same category for one pixel. For example, if the membership value of a pixel is 0.7 for forest at time *A*, and the membership value of that pixel is 0.2 for forest at time *B*, then the membership difference can be explained as a 50% decrease in the degree of forest membership of that pixel between times *A* and *B*. In this way, we will create *n* fuzzy comparison maps if there are *n* land cover categories.

Figure 8.1 shows the above process. Figure 8.1(A) shows the fuzzy forest polygons on the old land cover map, and Figure 8.1(B) shows the forest polygons on the new land cover map.   Figure 8.1(C) shows the difference between two forests.

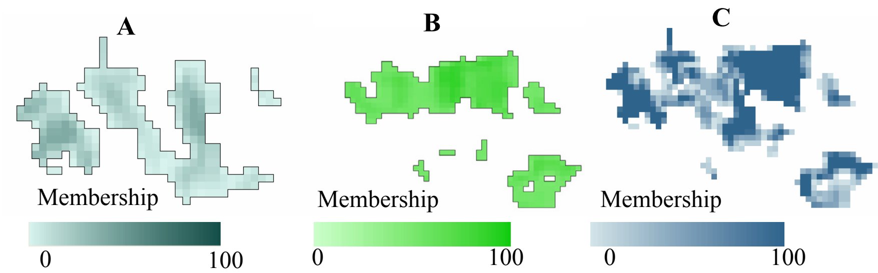


Figure 8.1 The membership difference between the old and new land cover objects

2. Reasoning about changes of fuzzy land covers based on categorical polygons

(1) Selection of reasoning methods
The purpose of a fuzzy inference system is to describe the change degree between land cover maps using linguistic variables that are represented by membership functions. Formally, a linguistic membership function is a mathematical curve that represents a person's intuitive perception of the degree of matching between sections of the input maps. By converting the linguistic expressions into membership functions, the change degree will be produced from the input maps.

Two fuzzy reasoning methods, namely the Mamdani and Tsukamoto methods, were described in Section 2.4. The Mamdani reasoning system is a rule-based decision model that produces mathematical control statements as output membership functions to handle the interactions of the inputs to the system. The general process is described in Figure 2.8. The design of this system requires a developer to create both input and output membership functions from linguistic interpretations of a subject. Through the compositional rule of inference and a defuzzification algorithm, Mamdani's system will produce an overall output value from the output membership functions. The advantage of Mamdani's fuzzy inference systems is that the fuzzy input and output membership functions are better suited to handling fuzziness and data uncertainty and work better with human input. A disadvantage is that the defuzzification process is computation-intensive and not easily subjected to rigorous quantitative analysis.

Unlike the Mamdani system, the Tsukamoto system does not contain output fuzzy membership functions. The individual crisp output is calculated based on the fuzzy equations and the final output is the weighted average of the individual crisp outputs. This system cannot propagate fuzziness from the input to outputs in an appropriate manner. For our purpose, the Mamdani method is selected for reasoning about change degrees of land covers.

(2) Average of membership values of polygons
If the classifications and membership values, as well as the registration are all error-free, then the comparison can be made directly by the differences in these *n* land cover categories. However, all these aspects contain errors. In order to minimize the effect of misregistration between two land cover maps, the polygon matching method is better than the pixel-by-pixel method (Power 2001). Power adopted the polygons that are derived from the intersection between the old land cover map and the new land cover map as the minimal unit for the reasoning. The intersection ratio between the new polygon and the old polygon is applied as the data source for fuzzy reasoning.

For the same reason, we adopt the polygons for the reasoning. The fuzzy polygons are identified by the intersection between the old land cover map and the new land cover map. The polygons are based on the new land cover map. Figure 8.2 shows the intersection polygons.

By the intersection of each polygon with the comparison maps, we will get a set of membership difference values for each polygon. These membership differences are then averaged as the attribute of each polygon.

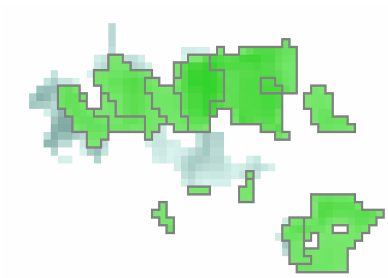


Figure 8.2 Generation of polygons for reasoning

(3) Creation of membership functions
The creation of input membership functions depends on the development of a linguistic scaling of the change degree of the polygons from the averaged membership values of these polygons. Semantic expressions are needed as answers to the question: "To what degree has the land cover changed for a specific polygon?" A five point scale is generated, ranging from tiny change to huge change in the membership difference of polygons. The meaning of these scaling values is as follows:

Tiny: the average of membership differences is very small;
Small: the average of membership differences is small;
Medium: the average of membership differences is medium;
Large: the average of membership differences is large;
Huge: the average of membership differences is very large.

Five linguistic variables are designed to represent the average membership differences: *tiny, small, medium, large* and *huge* (Figure 8.3). Three trapezoid curves are adopted as the membership functions to represent the linguistic variables *small, medium* and *large* of the average of membership difference. A decreasing half-trapezoid curve is used as the membership function to represent the linguistic variable *tiny*, and an increasing half-trapezoid curve for the linguistic variable *huge*. These membership functions match human intuition about the magnitude of difference. If the difference is less than a small value, then the difference is tiny. If the difference is greater than a large number, then the difference is very large. The transition between these linguistic variables is smooth.

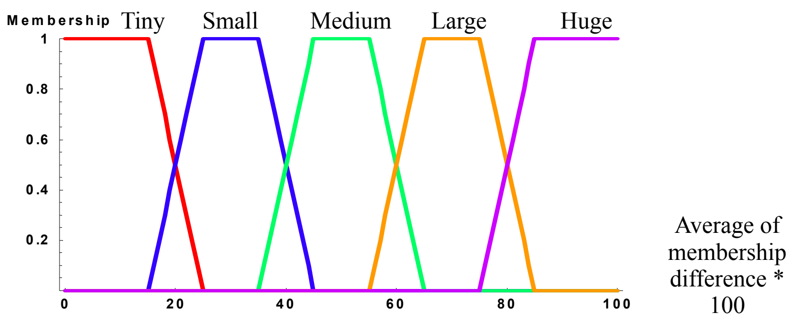


Figure 8.3 Membership functions of input variables (membership difference)

Similar membership functions are adopted to represent the linguistic variables for the polygon area, which is converted to pixel numbers. Following Power's idea, three linguistic variables are designed to represent the pixel numbers *few, normal* and *many*. The parameters of these membership functions are illustrated in Figure 8.4. If the pixel number is less than 2*2 pixels, people normally regard the area is small. Sometimes, the size of four pixels is taken as the smallest mapping unit in many applications. The membership function is designed by a decreasing half-trapezoid curve. If the pixel number is between 2*2 and 4*4 pixels, then the pixel number is regarded as normal size, which is represented by a trapezoid curve. If the pixel number is greater than 3*3, then the pixel number is many, represented by an increasing half-trapezoid curve.

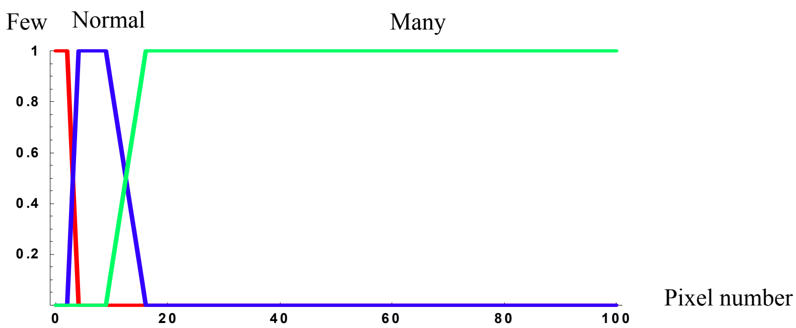


Figure 8.4 Membership functions of input variables (pixel number)

For the output, five linguistic variables are designed to represent the change degree for the categorical land covers: *tiny, small, medium, large* and *huge* (Figure 8.5).

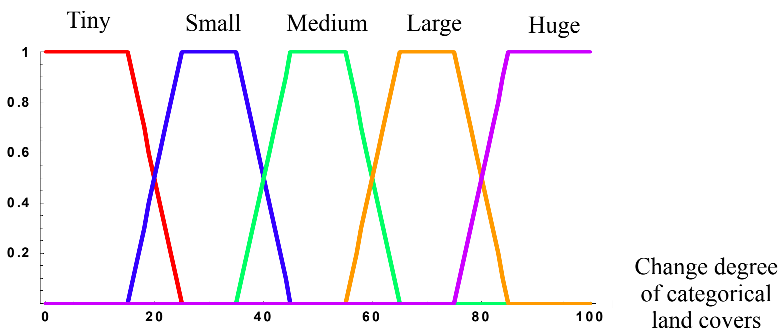


Figure 8.5 Membership functions of output variables (change degree of the categorical land covers)

(4) Inference rules

The essential part of a fuzzy inference system is a set of fuzzy rules that are related by means of a fuzzy implication function and a compositional rule of inference, as introduced in Chapter 2. Fuzzy rules are a collection of linguistic IF-THEN statements that describe how a fuzzy inference system makes a decision about categorizing an input or controlling an output.

In Mamdani's reasoning method, the fuzzy rules are represented by a Mamdani implication function. The implication process defines the associations between the input membership functions and determines the consequence of a rule. Furthermore, the fuzzy implication of a rule depends on its IF-THEN connective operator, which expresses how a fuzzy rule is delineated by a fuzzy relation. The premise variables *a* and *b* of the rules are connected by the minimum operator *min(a,b)*. The fuzzy rules are expressed as:

IF a and b, THEN c.

Fifteen rules are designed subjectively for the fuzzy reasoning. In order to minimize the effect of misregistration error, the result of these inference rules changes transitionally based on the combination of membership difference and pixel numbers.

They are as follows:

(1) If the membership difference is tiny, and the pixel number is few, then the change degree is tiny;
(2) If the membership difference is small, and the pixel number is few, then the change degree is tiny;
(3) If the membership difference is medium, and the pixel number is few, then the change degree is tiny;
(4) If the membership difference is large, and the pixel number is few, then the change degree is small;
(5) If the membership difference is huge, and the pixel number is few, then the change degree is medium;
(6) If the membership difference is tiny, and the pixel number is normal, then the change degree is tiny;
(7) If the membership difference is small, and the pixel number is normal, then the change degree is small;
(8) If the membership difference is medium, and the pixel number is normal, then the change degree is medium;
(9) If the membership difference is large, and the pixel number is normal, then the change degree is medium;
(10) If the membership difference is huge, and the pixel number is normal, then the change degree is large;
(11) If the membership difference is tiny, and the pixel number is many, then the change degree is tiny;
(12) If the membership difference is small, and the pixel number is many, then the change degree is small;
(13) If the membership difference is medium, and the pixel number is many, then the change degree is medium;
(14) If the membership difference is large, and the pixel number is many, then the change degree is large;
(15) If the membership difference is huge, and the pixel number is many, then the change degree is huge.

(5) Defuzzification
To obtain a crisp change degree value, it is necessary to transform the output

membership functions produced by the inference system algorithm into a crisp number. There are numerous methods for the defuzzification process, and two of them were introduced in Chapter 2. For the defuzzification, the central area method is adopted.

(6) Composition of results of fuzzy reasoning
The reasoning is individually processed for each category of fuzzy land covers. Therefore we will get *n* results of change degree for categorical land covers. Since the polygons may overlap with each other, the result is also overlapped. That is, we will get two results for each pixel, showing the change degree. For example, the change degree for forest is illustrated in Figure 8.6(A), and the change degree for bush in Figure 8.6(B).

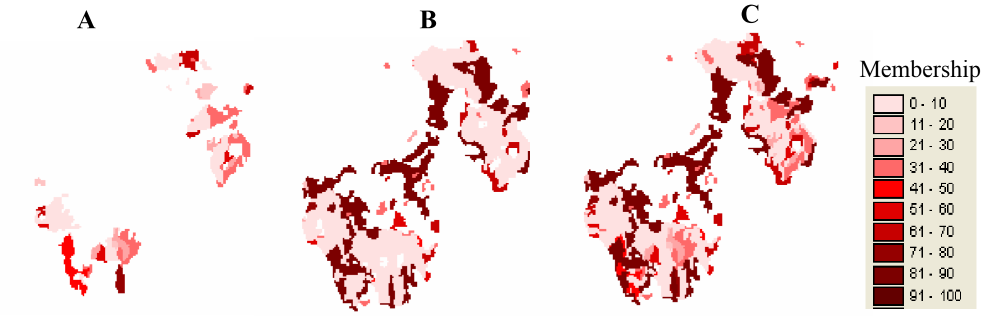


Figure 8.6 Composition of change degree for different land covers

In order to know the change degree for each pixel, we have to make a composition of these results. The composition is calculated by the sum of change degree of each land cover for each pixel. That is, supposing there are *n* fuzzy land cover categories, then the change degree is:

$$CD = \sum_{i=1}^{n} z_i$$

where *CD* is the crisp value representing the categorical change degree, and z_i is the crisp value of the change degree for each land cover. Figure 8.3(C) shows the composition of change degree for bush and forest.

3. Reasoning about change degree of land covers

The result after the above reasoning will show the change degree of land cover objects based on categorical polygons. If the categories are correctly classified and membership values at each pixel are precisely calculated, then the above result is able to show the change degree of land covers, avoiding the misregistration errors. However, as we know, there are always errors in image classification. The fuzzy land covers are derived based on maximum likelihood classifiers. The fuzzy land cover objects contain errors in categories as well as membership values. In order to minimize the errors in fuzzy land cover objects, the change degree of a land cover object is calculated based on the combination of the spectral value changes and the above category change results.

The input linguistic variables are the categorical change degree and the spectral value differences. The polygonal change degree is represented by five scaling variables, whose membership functions are the same as the membership functions described in Figure 8.5.

The values of spectral changes are derived based on the comparisons between two images. Figure 8.7 shows the values.

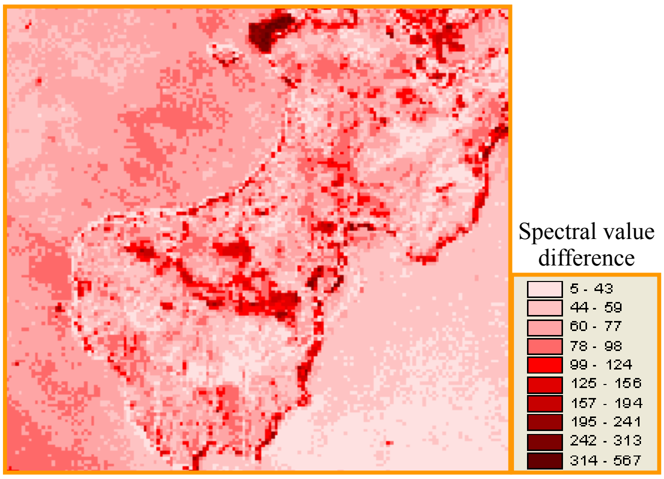


Figure 8.7 Spectral value differences

The input of the spectral value change for the fuzzy reasoning is represented by five linguistic variables: *tiny, small, medium, large* and *huge*. The meaning of these variables is explained as:

Tiny: the spectral value change is tiny;
Small: the spectral value change is small;
Medium: the spectral value change is medium;
Large: the spectral value change is large;
Huge: the spectral value change is very large.

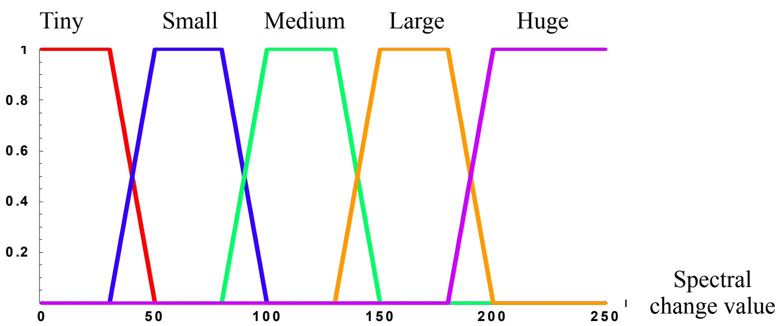


Figure 8.8 Membership functions of input variables (spectral value change)

The parameters of the membership functions of these five variables are illustrated in Figure 8.8.

The reasoning rules are designed as follows:

(1) If the spectral value change is tiny, and the categorical change is tiny, then the change degree is tiny;
(2) If the spectral value change is small, and the categorical change is tiny, then the change degree is tiny;
(3) If the spectral value change is medium, and the categorical change is tiny, then the change degree is tiny;
(4) If the spectral value change is large, and the categorical change is tiny, then the change degree is small;
(5) If the spectral value change is huge, and the categorical change is tiny, then the change degree is small;
(6) If the spectral value change is tiny, and the categorical change is small, then the change degree is tiny;
(7) If the spectral value change is small, and the categorical change is small, then the change degree is small;
(8) If the spectral value change is medium, and the categorical change is small, then the change degree is small;
(9) If the spectral value change is large, and the categorical change is small, then the change degree is medium;
(10) If the spectral value change is huge, and the categorical change is small, then the change degree is medium;
(11) If the spectral value change is tiny, and the categorical change is medium, then the change degree is tiny;
(12) If the spectral value change is small, and the categorical change is medium, then the change degree is small;
(13) If the spectral value change is medium, and the categorical change is medium, then the change degree is medium;
(14) If the spectral value change is large, and the categorical change is medium, then the change degree is large;
(15) If the spectral value change is huge, and the categorical change is medium, then the change degree is large;
(16) If the spectral value change is tiny, and the categorical change is large, then the change degree is small;
(17) If the spectral value change is small, and the categorical change is large, then the change degree is medium;
(18) If the spectral value change is medium, and the categorical change is large, then the change degree is medium;
(19) If the spectral value change is large, and the categorical change is large, then the change degree is large;
(20) If the spectral value change is huge, and the categorical change is large, then the change degree is huge;
(21) If the spectral value change is tiny, and the categorical change is large, then the change degree is small;
(22) If the spectral value change is small, and the categorical change is large, then the change degree is medium;
(23) If the spectral value change is medium, and the categorical change is large, then the change degree is large;
(24) If the spectral value change is large, and the categorical change is large, then the change degree is large;

(25) If the spectral value change is huge, and the categorical change is large, then the change degree is huge;

Based on these inference rules, the change degree can be inferred. The final crisp value is defuzzified based on the center of area. It can be regarded as the final result of the change degree between two land cover maps.

8.4 Test area

Sanya city is selected as the test area. The city is located on Hainan Island, in the south of China. The TM images were obtained on 18 April 1990 (A) and 27 October 1998 (B). Figure 8.9 shows about 220*200 pixels.

A B

Figure 8.9 Two TM images

The registration is done by ARC/INFO. The mean average error (RMS) of the registration between two images is 0.8.

The bi-temporal images were classified into 11 categories of land cover objects. The categories were depicted in Chapter 6. The categories are then merged into eight categories, where the paddy field and dry land are merged into arable land, rural land and built-up area are merged into residential area, and waste land and bare land are grouped into waste land.

The eight classes are as follows:

(1) Forest: in a pixel most trees are greater than 6 m with the canopy generally covering over 80% of the pixel;

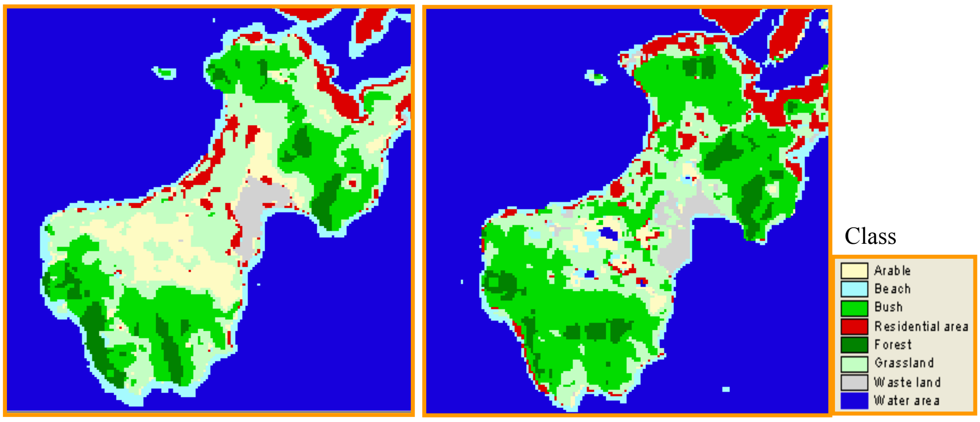
(2) Bush: in a pixel most trees are between 2 and 6 m with the canopy generally covering over 50% to 80% of the pixel;

(3) Shrub and grassland: in a pixel there are some trees normally less than 2 m with the canopy generally covering between 50 and 80% of the pixel;

(4) Waste land: in a pixel there are some trees less than 1 m with the canopy

generally covering between 0 and 50% of the pixel;
(5) Water body: a pixel covered by water;
(6) Beach: a pixel covered by wet sands and some water;
(7) Arable land: which includes paddy field and dry land;
(8) Residential area: most of the pixel covered by buildings, roads or other construction material.

The dominant layer of two fuzzy land cover maps is illustrated in Figure 8.10.

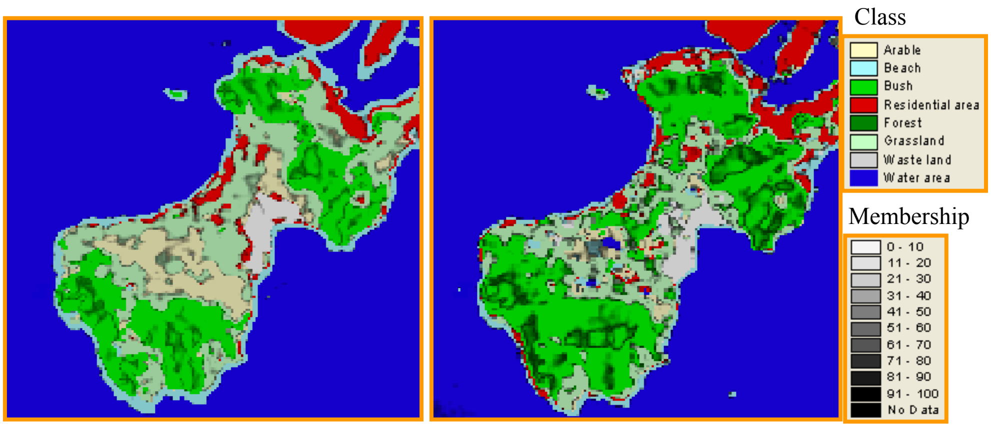


A: Crisp land cover objects in the old image

B: Crisp land cover objects in the new image

Figure 8.10 Crisp representation of land cover objects

Figure 8.11 shows the dominant layer of two fuzzy land objects with membership values.



A: Fuzzy land cover objects in the old image

B: Fuzzy land cover objects in the new image

Figure 8.11 Fuzzy representation of land cover objects (dominant objects)

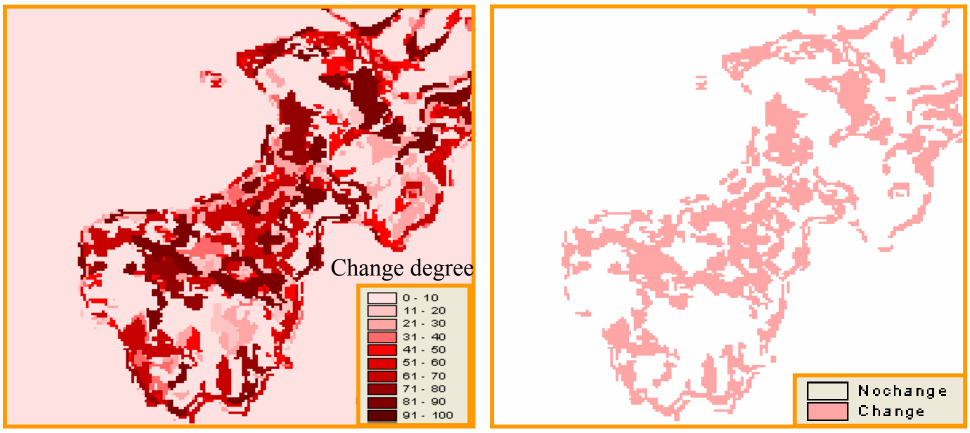
For the old land cover map, the accuracy of the classification is 88% for the dominant class and 75% for the secondary class; the overall accuracy is 67%. For the new land cover map, the accuracy of the classification is 80% for the dominant class and 70% for the secondary class; the overall accuracy is 60%.

8.5 Results

8.5.1 Results

(1) Results of change degree based on categorical polygons

The results of change degree of land covers based on categorical polygons are shown in Figure 8.12. Figure 8.12(A) shows the results of changes of land cover polygons. Figure 8.12(B) shows the crisp result of change in which the change value is greater than 60.



A. Fuzzy representation B. Crisp representation

Figure 8.12 Changes in categorical land cover polygons

Table 8.1 Changes of land cover polygons

Old_class	Total_pixel	Change_pixel	Percentage
Arable	1597	1318	83%
Beach	1667	1112	67%
Bush	3107	612	20%
Residential	1022	437	43%
Forest	871	422	49%
Grassland	3802	2404	63%
Waste land	402	151	38%
Water	24014	420	2%

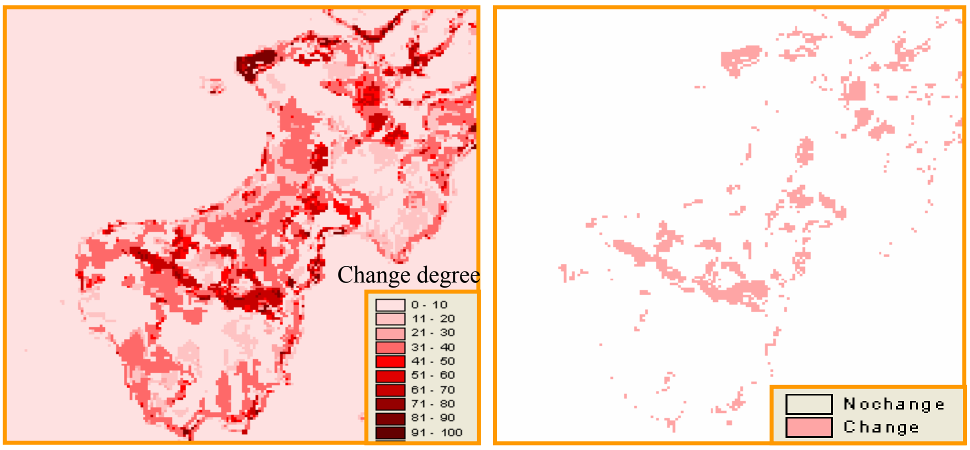
Because of the errors in land cover classifications, the change between land cover polygons is exaggerated in many places, although these places have almost no change in land covers. Table 8.1 shows these changes based on the old crisp land cover polygons (Figure 8.10(A)). In Table 8.1, the residential area shows a 43% change into other land covers. However, this is almost impossible in reality.

Table 8.2 shows the details of residential areas changes. 23% of residential areas have changed into grassland, most of which is wrong. This is because these residential areas are covered by many individual trees in the new image and therefore wrongly classified into grassland.

Table 8.2 Changes of residential polygons

Old_class	New_class	Total_pixel	Pixel_number	Value
Residential	Arable	1022	8	0.78%
Residential	Beach	1022	96	9.39%
Residential	Bush	1022	52	5.09%
Residential	Residential	1022	585	57.24%
Residential	Forest	1022	0	0.00%
Residential	Grassland	1022	236	23.09%
Residential	Waste land	1022	43	4.21%
Residential	Water	1022	2	0.20%

(2) Results of reasoning about change degree of land covers



A. Fuzzy representation B. Crisp representation

Figure 8.13 Changes of land covers

Figure 8.13 shows the results of change degree of land covers. Errors due to the wrong classifications are much decreased. Figure 8.13(A) shows the fuzzy representation of changes of land covers based on the new land cover map. Figure 8.13(B) shows the crisp result of change in which the change value is greater than 60.

Table 8.3 shows the improvement made by the second-step reasoning. The changes from residential area into others are much decreased. In total only 5.5% has changed into other land covers. Compared with 57.24% in Table 8.2, this is a decrease of almost 52%.

Table 8.3 Changes of residential area based on the old land cover map

Old_class	New_class	Total_pixel	Adjusting	Pixel	Change_ Percentage
Residential	Arable	1022	No change	8	
Residential	Beach	1022	No change	81	
Residential	Beach	1022	Changed	15	1.47%
Residential	Bush	1022	No change	49	
Residential	Bush	1022	Changed	3	0.29%
Residential	Residential	1022	No change	581	
Residential	Residential	1022	Changed	4	0.39%
Residential	Grassland	1022	No change	215	
Residential	Grassland	1022	Changed	21	2.05%
Residential	Waste	1022	No change	32	
Residential	Waste	1022	Changed	11	1.08%
Residential	Water	1022	Changed	2	0.20%

Table 8.4 Changes of land covers based on the old land cover map

Old_class	Total_pixel	Change_pixel	Percentage
Arable	1597	650	40.7%
Beach	1667	332	19.9%
Bush	3107	38	1.2%
Residential	1022	56	5.5%
Forest	871	1	0.1%
Grassland	3802	288	7.6%
Waste land	402	100	24.9%
Water	24014	224	0.9%

Table 8.4 shows the changes of land covers based on old land cover maps. It shows that the arable land changes a lot over eight years. Forest, bush and water area show almost no changes. Actually these three land covers are very stable in Sanya city. Nearly 25% of waste land changes into other land covers. In many areas, beach and waste land have changed into grassland. 5.5% of residential area has changed into other land covers.

Table 8.5 shows the changes based on the new land cover map, and the net change percentages. $Percentage_1$ is the increase percentage of the land cover from other land covers. $Percentage_2$ is the decrease percentage of the land cover into others. The net change is $percentage_1$ minus $percentage_2$. This shows that over eight years arable land decreases nearly 30%, and grassland increases 13.5%. The residential area increases 13.5%, and bush increases nearly 5%.

Table 8.5 Changes of land covers

New_class	Total_ Pixel	Change_ pixel	$Percentage_1$	$Percentage_2$	Net change
Arable	479	54	11.0%	40.7%	-29.7%
Beach	1184	206	17.0%	19.9%	-2.9%
Bush	5025	288	6.0%	1.2%	4.8%
Residential	1512	282	19.0%	5.5%	13.5%
Forest	908	2	0.0%	0.1%	-0.1%
Grassland	2907	612	21.0%	7.6%	13.4%
Waste land	507	132	26.0%	24.9%	1.1%
Water	23960	113	0.0%	0.9%	-0.9%

8.5.2 Comparisons

The above method calculates the changes of land covers by a two-step reasoning method. If the land covers are crisply represented, the changes between land covers can be calculated directly based on the difference between the old and new land cover polygons. Figure 8.14 shows the results of changes based on land cover polygons. It is unrealistic since in reality there is not such a big change.

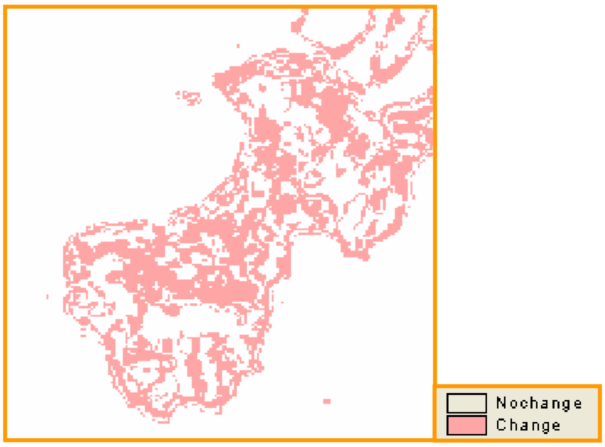
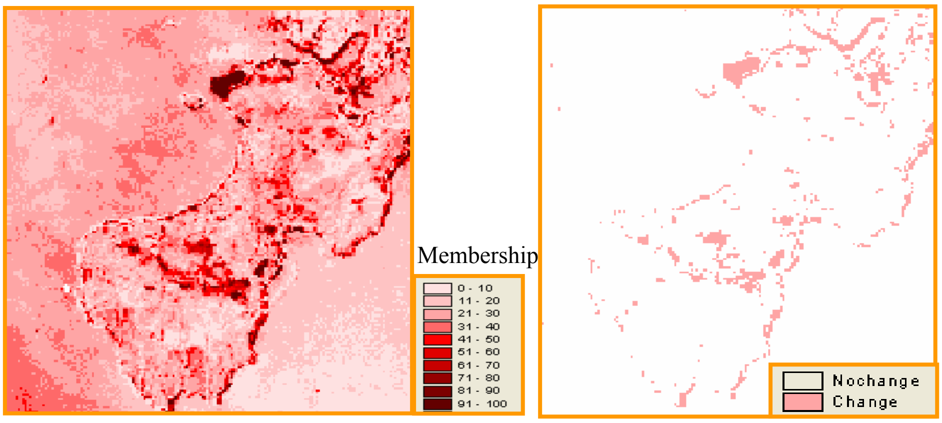


Figure 8.14 Crisp changes of categorical land cover polygons

We can adjust these changes also based on the spectral differences between two images. For example, we can assume that if the spectral difference is greater than 180, then there is a change; and if the spectral difference is less than 25, then there is no change; between 25 and 180, there is a transitional change. Then we will also derive a change map of land covers. However, since the spectral value difference does not mean changes of land covers, it will show that some water areas that actually are unchanged have changed. Figure 8.15 shows the land cover changes based on this method. The left-hand lower corner is the sea area, but since there is a difference in spectral values, it also shows changes. If we calculate changes at 0.5-level, Figure 8.15(B) can be derived. However, many changes are neglected, for example changes from grassland to residential area.



A. Fuzzy representation B. Crisp representation

Figure 8.15 Changes of land covers considering spectral value changes

8.5.3 Transitional changes of land covers

The above comparison shows that adopting the fuzzy reasoning method achieves better results than traditional crisp methods. One of the biggest advantages of using fuzzy reasoning is that not only can the changes be calculated, but the transitional changes can also be detected. It should be mentioned that forest or bush regeneration and succession are complex and complicated processes that are often difficult to model with traditional Boolean techniques. This is partly due to the inability of such techniques to represent the intermediate growth patterns. The changes of bush are illustrated in Figure 8.16. Although the change of bush is 6% based on the new land cover map, there are many transitional changes from bush to others and others to bush.

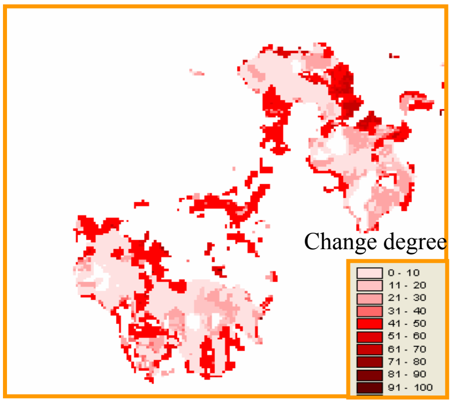


Figure 8.16 Transitional changes of bush

8.6 Conclusions and discussions

The comparison of land cover maps is the basis for many dynamic analysis of land use and land covers. The traditional method usually compares the differences based on a crisp pixel-by-pixel method. These Boolean similarity operations often cannot adequately account for the errors and complexity inherent in spatial information. A fuzzy method may mitigate these difficulties.

In this chapter, a two-step reasoning method is proposed for measuring both similarities and land cover changes between maps while accounting for the errors in the data sets. It has been shown that this method is less affected by misregistration errors and classification errors. The first-step reasoning largely solves the misregistration errors because the comparison is done based on fuzzy polygons. In order to solve the misclassification errors, the fuzzy reasoning is made by combining spectral value differences with the results of the first-step reasoning. Furthermore, the transitional changes of land covers can be detected since rich information can be derived by the use of fuzzy reasoning.

In this method, several issues should be mentioned. Firstly, the first-step reasoning will minimize the misregistration errors; however, in the second-step reasoning, the misregistration error is re-imported because of misregistration of two images. Possibly a better method is to integrate two reasoning steps into one step. Secondly, the classification errors are minimized by combining the spectral value differences. The spectral value difference is actually the distance sum between each band of images. This data source can be replaced by other direct results from two images; for example, we can also measure the angles between the bands of two images. Thirdly, it should be mentioned that no matter what data sources are adopted to improve the change calculation, these data also contain errors besides the misregistration. This method cannot remove the errors, but it will greatly decrease them. Fourthly, the fuzzy reasoning rules are designed based on practical knowledge. It is also possible to adopt a fuzzy neural network approach to improve the fuzzy reasoning rules. The fuzzy neural network approach will greatly reduce the subjective effect of humans. Finally, more investigation should be carried out into spatial correlation. The spatial correlation between the land cover maps derived from two images is certain if no land cover changes totally. Spatial correlation can also improve the calculation of changes caused by misregistration and misclassification errors, since it measures the patterns of land covers.

This chapter provides the way for adopting fuzzy land cover objects to improve the understanding of land cover changes. The result shows that fuzzy spatial objects can be used not only to achieve more precise results, but will also to derive more information from spatial objects.

Chapter Nine

Conclusions and Discussions

9.1 Summary

Currently most GISs represent natural phenomena by crisp spatial objects. In fact many natural phenomena have fuzzy characteristics. The representation of these objects in the crisp form greatly simplifies the handling methods for GIS and can still achieve useful results for many applications. However, this simplification cannot precisely describe these natural phenomena, and it will cause loss of information in these objects. In order to describe natural phenomena more precisely, the fuzziness in these natural phenomena should be described and represented in GIS to derive better results and for a better understanding of the real world.

The central topic of the whole thesis focuses on fuzzy spatial objects for accommodation in a GIS. Several issues are discussed theoretically and practically, including defining fuzzy spatial objects, the topological relations between them, modeling fuzzy spatial objects, generating fuzzy spatial objects and the advantage of using fuzzy spatial objects for certain applications.

A formal definition of spatial objects is usually derived based on highly abstract mathematics such as set theory and topology. Fuzzy set theory and fuzzy topology are the ideal tools for defining fuzzy spatial objects theoretically. In general, fuzzy set theory is a natural extension of classic set theory, such that an object is not a set of elements either true or false, but it is a set of elements with a degree of membership in the range 0 to 1. Fuzzy topology is also a natural extension of ordinary topology that is built up based on fuzzy sets. Since a crisp set is a special form of fuzzy set, the ordinary topology is also a special form in fuzzy topology in which the crisp set is just a special case. However, because of the extension, several properties holding between crisp sets do not hold for fuzzy sets. For example, the excluded-middle law does not hold between fuzzy sets. These are the same in fuzzy topology. For example, the neighborhoods of a fuzzy set cannot be adopted for defining fuzzy topology. On the other hand, many notions should be extended for fuzzy set theory as well as for fuzzy topology. The same notions may have different meanings. For these notions the best link between the ordinary topology and fuzzy topology is that the fuzzy topological space of a universe is induced from the ordinary (crisp) topological space. Many properties of a crisp topological space still hold in the induced fuzzy topological space.

In order to define fuzzy spatial objects, several notions are revisited around the definition of fuzzy boundary in fuzzy topology. Three definitions of fuzzy boundary are analyzed and one is selected for defining fuzzy spatial objects. Besides the fuzzy boundary, several notions such as the core, the fringe, the frontier, the internal fringe, the external fringe and the outer of a fuzzy set are defined. The relationships between these notions and the interior, the boundary and the exterior of a fuzzy set are revealed. In general, the core tells a finer structure than the interior, and the fringe shows a finer structure than the boundary of a fuzzy set in fuzzy topological space. These concepts are all proven to be topological properties of a fuzzy topological space.

The definition of a simple fuzzy region is derived based on the topological properties. It has been discussed twice in the thesis. Firstly, the definition of a simple fuzzy region is given in a special fuzzy topological space called crisp fuzzy topological space, since most topological properties of a fuzzy set in the fuzzy topological space are the same as those in crisp topological space. A formal definition of a simple fuzzy region is proposed based on the discussion of the topological properties besides the interior, the boundary and the exterior of a fuzzy set in the general fuzzy topological space. A crisp simple region is a special form of a simple fuzzy region.

One of the fundamental aspects of fuzzy spatial objects is the topological relations. This topic is intensively discussed in the thesis. The problem using the 9-intersection approach for the identification of topological relations between fuzzy spatial objects is revealed. In order to derive the topological relations between fuzzy spatial objects, the 9-intersection approach is updated into the 3*3-intersection approach in the crisp fuzzy topological space. Furthermore, the 4*4-intersection matrix is built up by using the topological properties of fuzzy sets, and the 5*5-intersection matrix can be built up based on a certain condition in crisp fuzzy topological space. These matrices are then updated in the general fuzzy topological space by using other topological properties of fuzzy sets. Two 3*3-intersection and one 4*4-intersection matrices are presented in the general fuzzy topological space. The topological relations between simple fuzzy regions are then identified based on the topological invariants in the intersections of the intersection matrices. Forty-four (44) and 152 relations are identified between two simple fuzzy regions by using the empty/non-empty topological invariants in the intersections.

The modeling of fuzzy spatial objects should be done not only for simple fuzzy regions, but also for fuzzy lines and fuzzy points. In order to model fuzzy lines and fuzzy points and the topological relations between fuzzy spatial objects, a fuzzy cell complex is proposed based on fuzzy cells. Fuzzy region, fuzzy line and fuzzy point are then defined according to this structure. The relations between these fuzzy spatial objects are then identified. The fuzzy cell complex structure can easily model the fuzzy spatial objects. It constitutes the theoretic framework for modeling fuzzy spatial objects.

After dealing with the theoretic framework for modeling fuzzy spatial objects, the thesis addresses several practical issues of applying fuzzy spatial objects. The first issue is how to generate fuzzy spatial objects. A composite method is proposed for generating fuzzy land cover objects. It involves several steps, from the design of membership functions to classification and refining the membership values of fuzzy land cover objects.

Another practical issue is how to query fuzzy spatial objects, particularly based on topological relations. In traditional GIS, the query operators are defined based on the relatively small number of topological relations. However, there are many topological relations between two fuzzy spatial objects. In order to query fuzzy spatial objects, the query operators are formed based on the six common-sense operators in traditional GIS. Then the 44 or 152 topological relations are grouped into these operators by four different methods. These methods constitute the query methods to meet the different requirements of applications.

The third practical issue is how to use fuzzy spatial objects in real applications. Since the dynamics of land covers is a very important topic in China, the focus is on calculating changes of land covers. Sanya city is selected as the test area. A fuzzy reasoning method is discussed for calculating the changes. It shows that, with the fuzzy objects, not only can the changes be calculated, but also the details of the changes can be revealed.

9.2 Conclusions

Concerning the research questions, the following conclusions can be drawn:

(1) Fuzzy set theory and fuzzy topology are the ideal mathematical tools for defining fuzzy spatial objects. There are many topological properties for a fuzzy set in fuzzy topological space that can be adopted for the formal definition of fuzzy spatial objects. A simple fuzzy region, fuzzy region, simple fuzzy line, fuzzy line and fuzzy point are formally defined based on the topological properties of a fuzzy set in fuzzy topological space.
(2) Topological relations should be identified based on the topological properties of fuzzy spatial objects. Two forms of a 3*3-intersection matrix and one 4*4-intersection matrix are introduced using different properties of fuzzy sets. These matrices can be adopted for identifying topological relations between fuzzy spatial objects. Besides 44 topological relations, 152 relations are identified between two simple fuzzy regions.
(3) A theoretic framework for modeling fuzzy spatial objects can be built based on the fuzzy cell complex structure. The structure is able to represent fuzzy points, lines, and regions. The topological relations between different fuzzy spatial objects can be identified based on intersection matrices.
(4) There are many methods for generating fuzzy spatial objects. The composite method is one of the methods that can be used to generate fuzzy spatial objects.
(5) The query of fuzzy spatial objects in GIS should be simple and easily understandable. Different applications may have different requirements. The

four different query methods in the thesis can meet different requirements in a relatively complete way, yet they are all derived based on a strong theoretic background.

(6) The utilization of fuzzy spatial objects is decided by the applications. Fuzzy spatial objects are absolutely suitable for analyzing most natural phenomena since they have fuzzy characteristics. One of the advantages of adopting fuzzy spatial objects lies in the fact that many details can be revealed.

Concerning the objectives, the thesis has provided a formal definition of fuzzy spatial objects, several intersection approaches for identifying the topological relations between fuzzy spatial objects, a theoretic framework for modeling fuzzy spatial objects, a method for generating fuzzy spatial objects, and several query methods for different requirements; moreover, it shows the capability of fuzzy spatial objects to improve understanding of land cover changes. It is better to represent natural phenomena in the fuzzy form since it can provide more information than the crisp form.

9.3 Contributions

The main contribution of the thesis is the analysis of topological relations between fuzzy spatial objects. It can be broken down into several aspects:

(1) There are more topological properties of a fuzzy set in fuzzy topological space. The core, the fringe, the frontier, the internal fringe, the external fringe, the outer, etc. are all topological properties. The analysis of these notions will help us to recognize the structure of fuzzy topology in more detail.

(2) Fuzzy spatial objects are formally defined. The definition includes fuzzy simple region, fuzzy region, fuzzy simple line, fuzzy line, and fuzzy point. Since these definitions are strictly derived from the properties of a fuzzy set in fuzzy topological space, they will greatly help modeling fuzzy spatial objects in practice.

(3) Methods for identifying the topological relations between two fuzzy spatial objects are presented. The different forms of the 3*3-intersection matrices, the 4*4-intersection matrix and the 5*5-intersection matrix are formalized for the identification of topological relations. Forty-four (44) topological relations are identified between two simple regions based on a 3*3-intersection matrix, and 152 relations are also identified based on the 4*4-intersection matrix.

(4) The fuzzy cell complex structure is proposed based on fuzzy cells. It can be applied to model fuzzy spatial objects, including fuzzy points, fuzzy lines and fuzzy regions. The topological relations between different kinds of fuzzy spatial objects are systematically analyzed and identified.

Other contributions include:

(1) A composite method for generating fuzzy spatial objects. The method is verified by generating fuzzy land cover objects from TM images.

(2) Four different methods are proposed for querying fuzzy spatial objects based on topological relations. These methods share the same query operators, which are simple and easily understood.

(3) A reasoning method is proposed for calculating changes of land covers. This method will reduce the uncertainties in data sets.

9.4 Discussions

Many issues should be discussed.

(1) The topological properties including the core, the fringe, the internal fringe, the frontier, the outer etc., of a fuzzy set in fuzzy topological space should be systematically analyzed. In the thesis a general analysis has been done to reveal the links between these properties and the interior, the boundary, the closure and the exterior of a fuzzy set. It should be pointed out that the analysis is just used to form fuzzy spatial objects. More research is needed on fuzzy topology.

(2) The topological relations should be further investigated. The 3*3-, 4*4- and 5*5-intersection matrices are all derived based on topological properties. The 44 or 152 relations are all identified based on the crisp topological invariants such as empty/non-empty intersections, and four comparisons. These relations are all fuzzy topological but are qualitatively described. More research on fuzzy topological relations should be done in the future.

(3) The definition of a simple fuzzy region is proposed based on the general fuzzy topological space. However, the definition includes many limitations. Are there some methods to simplify the definition?

(4) A formal model is proposed for representing fuzzy spatial objects. The part of the implementation of this model in computers is done based on current GIS software. However, conventional GIS can not represent fuzzy data types such as fuzzy lines. The implementation of fuzzy regions is done in a raster data model. The boundary of a fuzzy region is stored in the vector data model. The real implementation for accommodating fuzzy spatial objects has not been fulfilled.

(5) A composite method for generating fuzzy spatial objects has been proposed, which is essentially based on the maximum likelihood classification. Many other methods are available, such as the fuzzy neural network approach. The fuzzy neural network can be trained by sample data and possibly derive better inference rules. This method can also be adopted for calculating changes of fuzzy land cover objects.

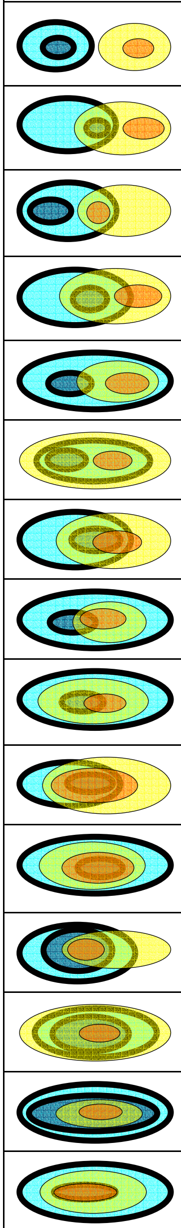
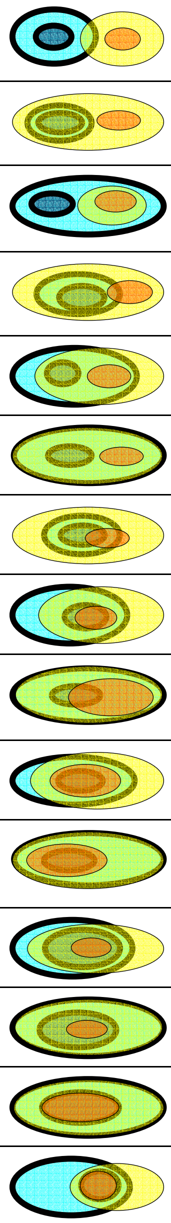
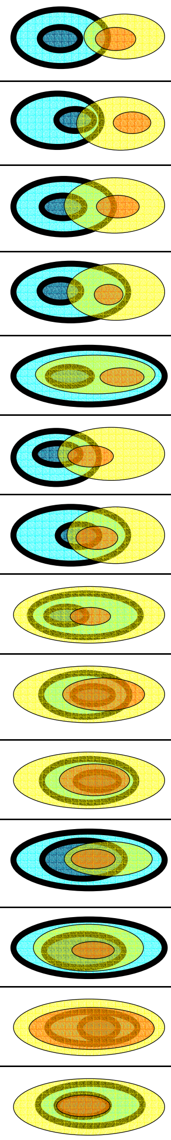
(6) The query methods are implemented based on ArcView GIS software. Since, as we mentioned, the real implementation of fuzzy spatial objects has not been done, the computing method for accessing fuzzy spatial objects is little discussed.

(7) The utilization of fuzzy spatial objects has demonstrated their power. However, a systematic analysis should be done on how many types of functions we can generate based on fuzzy spatial objects.

In short, the thesis provides a theoretic framework for modeling fuzzy spatial objects. More research should be done concerning the representation of fuzzy spatial objects at the data structure level. Practically, the thesis tackles some methods for generating fuzzy spatial objects. A GIS should be able to integrate different methods for generating fuzzy spatial objects from the real world. Finally, fuzzy spatial object functionality should be systematically analyzed and provided in GIS.

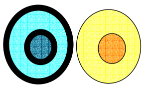
Appendix

Appendix 1. Forty-four (44) relations between two simple fuzzy regions by using the 3*3-intersection matrix (after Clementini and Di Felice 1996)

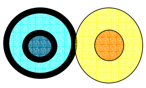
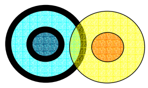
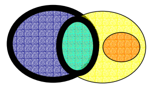
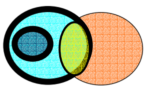
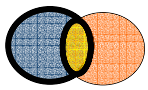
Illustration	Matrix	Illustration	Matrix	Illustration	Matrix
	(1) $\begin{bmatrix}0&0&1\\0&0&1\\1&1&1\end{bmatrix}$		(2) $\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$		(3) $\begin{bmatrix}0&1&1\\0&1&1\\1&1&1\end{bmatrix}$
	(4) $\begin{bmatrix}0&1&0\\0&1&1\\1&1&1\end{bmatrix}$		(5) $\begin{bmatrix}0&1&0\\0&1&0\\1&1&1\end{bmatrix}$		(6) $\begin{bmatrix}0&0&1\\1&1&1\\1&1&1\end{bmatrix}$
	(7) $\begin{bmatrix}0&0&1\\1&1&1\\0&1&1\end{bmatrix}$		(8) $\begin{bmatrix}0&0&1\\1&1&1\\0&0&1\end{bmatrix}$		(9) $\begin{bmatrix}0&1&1\\1&1&1\\1&1&1\end{bmatrix}$
	(10) $\begin{bmatrix}0&1&0\\1&1&1\\1&1&1\end{bmatrix}$		(11) $\begin{bmatrix}0&1&0\\1&1&0\\1&1&1\end{bmatrix}$		(12) $\begin{bmatrix}0&1&1\\1&1&1\\0&1&1\end{bmatrix}$
	(13) $\begin{bmatrix}0&1&1\\1&1&1\\0&0&1\end{bmatrix}$		(14) $\begin{bmatrix}0&1&0\\1&1&1\\0&1&1\end{bmatrix}$		(15) $\begin{bmatrix}0&1&0\\1&1&1\\0&0&1\end{bmatrix}$
	(16) $\begin{bmatrix}0&1&0\\1&1&0\\0&1&1\end{bmatrix}$		(17) $\begin{bmatrix}0&1&0\\1&1&0\\0&0&1\end{bmatrix}$		(18) $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$
	(19) $\begin{bmatrix}1&1&0\\1&1&1\\1&1&1\end{bmatrix}$		(20) $\begin{bmatrix}1&1&0\\1&1&0\\1&1&1\end{bmatrix}$		(21) $\begin{bmatrix}1&1&1\\1&1&1\\0&1&1\end{bmatrix}$
	(22) $\begin{bmatrix}1&1&1\\1&1&1\\0&0&1\end{bmatrix}$		(23) $\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}$		(24) $\begin{bmatrix}1&1&0\\1&1&0\\0&1&1\end{bmatrix}$
	(25) $\begin{bmatrix}1&1&0\\1&1&1\\0&0&1\end{bmatrix}$		(26) $\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$		(27) $\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$
	(28) $\begin{bmatrix}1&0&0\\1&1&1\\1&1&1\end{bmatrix}$		(29) $\begin{bmatrix}1&0&0\\1&1&1\\0&1&1\end{bmatrix}$		(30) $\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$
	(31) $\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix}$		(32) $\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$		(33) $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$
	(34) $\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$		(35) $\begin{bmatrix}1&1&0\\0&1&1\\0&1&1\end{bmatrix}$		(36) $\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$
	(37) $\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix}$		(38) $\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$		(39) $\begin{bmatrix}1&0&0\\1&0&0\\1&1&1\end{bmatrix}$
	(40) $\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}$		(41) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$		(42) $\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$
	(43) $\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$		(44) $\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$		

Appendix 2. One hundred and fifty-two (152) topological relations between two simple fuzzy regions in $(\widetilde{R}^2, C)$ by using the 4*4-intersection matrix

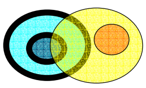
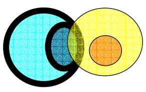
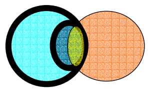
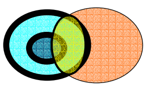
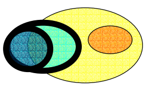
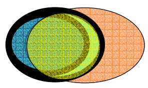
(1) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

	(1) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

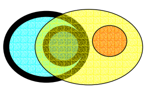
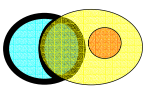
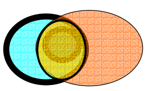
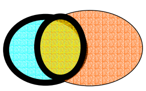
(2) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

	(2) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(3) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(4) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
	(5) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(6) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		

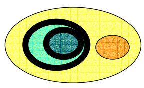
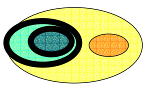
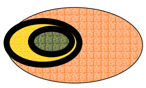
(3) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

	(7) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(8) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(9) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$
	(10) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(11) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(12) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

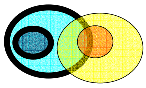
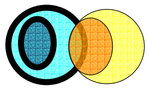
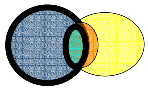
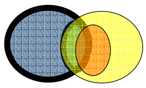
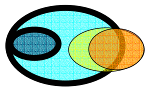
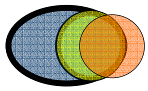
(4) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

	(13) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(14) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
	(15) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(16) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

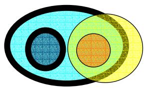
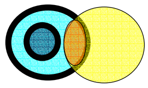
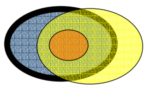
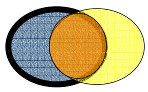
(5) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

	(17) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(18) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(19) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

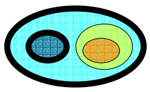
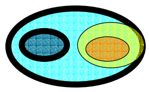
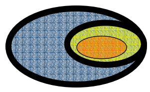
(6) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

	(20) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(21) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(22) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
	(23) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(24) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(25) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

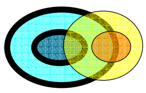
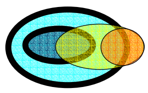
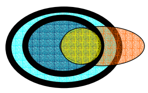
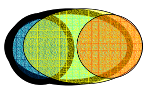
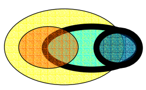
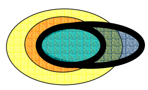
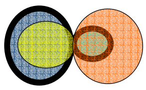
(7) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

	(26) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		(27) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$
	(28) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		(29) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

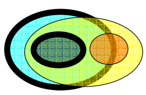
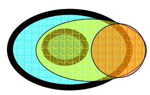
(8) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

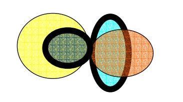
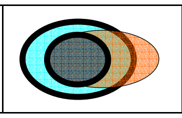
	(30) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$		(31) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$		(32) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(9) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

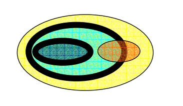
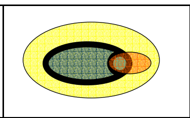
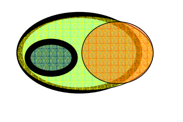
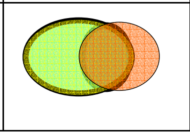
	(33) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(34) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(35) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$
	(36) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(37) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(38) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
	(39) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$				

(10) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

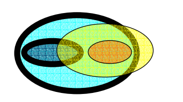
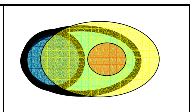
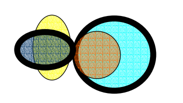
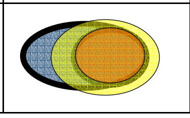
	(40) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(41) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

	(42) $\begin{bmatrix}0&0&1&0\\1&1&1&1\\1&1&0&1\\1&1&1&1\end{bmatrix}$		(43) $\begin{bmatrix}0&0&1&0\\1&1&0&1\\1&1&0&1\\1&1&0&1\end{bmatrix}$

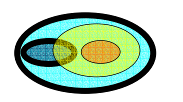
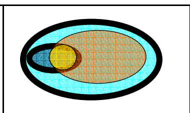
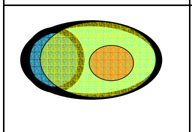
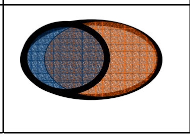
(11) $\begin{bmatrix}0&1&0\\1&1&0\\1&1&1\end{bmatrix}$

	(44) $\begin{bmatrix}0&0&1&0\\1&1&1&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$		(45) $\begin{bmatrix}0&0&1&0\\1&1&1&0\\1&0&0&0\\1&1&1&1\end{bmatrix}$
	(46) $\begin{bmatrix}0&0&1&0\\1&1&1&0\\1&1&1&0\\1&1&0&1\end{bmatrix}$		(47) $\begin{bmatrix}0&0&1&0\\1&1&0&0\\1&0&0&0\\1&1&0&1\end{bmatrix}$

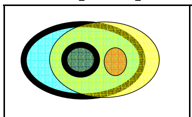
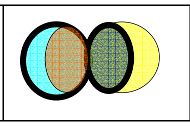
(12) $\begin{bmatrix}0&1&1\\1&1&1\\0&1&1\end{bmatrix}$

	(48) $\begin{bmatrix}0&1&1&1\\0&1&1&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$		(49) $\begin{bmatrix}0&1&1&1\\0&1&1&1\\1&1&1&0\\0&1&1&1\end{bmatrix}$
	(50) $\begin{bmatrix}0&1&1&1\\0&1&1&1\\1&1&0&1\\0&1&1&1\end{bmatrix}$		(51) $\begin{bmatrix}0&1&1&1\\0&1&1&1\\1&0&0&0\\0&1&1&1\end{bmatrix}$

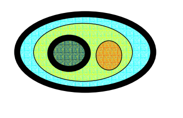
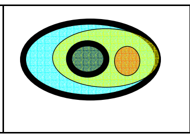
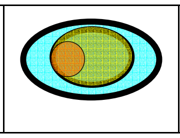
(13) $\begin{bmatrix}0&1&1\\1&1&1\\0&0&1\end{bmatrix}$

	(52) $\begin{bmatrix}0&1&1&1\\0&1&1&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$		(53) $\begin{bmatrix}0&1&1&1\\0&1&0&1\\1&1&0&1\\0&0&0&1\end{bmatrix}$
	(54) $\begin{bmatrix}0&1&1&1\\0&1&1&1\\1&1&1&0\\0&0&0&1\end{bmatrix}$		(55) $\begin{bmatrix}0&1&1&1\\0&1&0&1\\1&0&0&1\\0&0&0&1\end{bmatrix}$

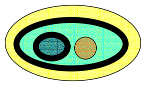
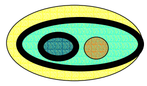
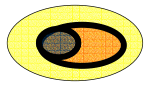
(14) $\begin{bmatrix}0&1&0\\1&1&1\\0&1&1\end{bmatrix}$

	(56) $\begin{bmatrix}0&0&1&0\\0&1&1&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$		(57) $\begin{bmatrix}0&0&1&0\\0&1&1&1\\1&1&0&1\\0&1&1&1\end{bmatrix}$

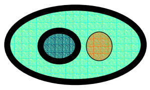
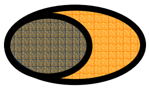
(15) $\begin{bmatrix}0&1&0\\1&1&1\\0&0&1\end{bmatrix}$

	(58) $\begin{bmatrix}0&0&1&0\\0&0&1&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$		(59) $\begin{bmatrix}0&0&1&0\\0&1&1&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$		(60) $\begin{bmatrix}0&0&1&0\\0&1&0&1\\1&1&0&1\\0&0&0&1\end{bmatrix}$

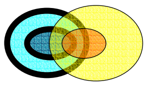
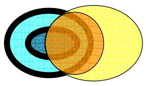
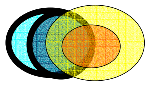
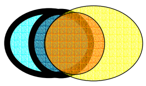
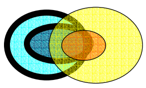
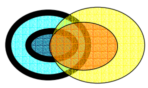
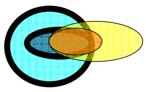
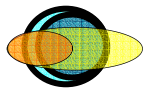
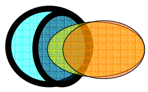
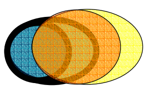
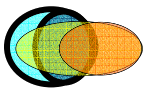
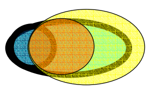
(16) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

	(61) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		(62) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		(63) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

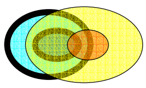
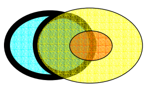
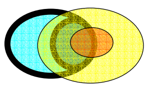
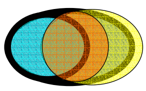
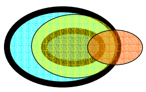
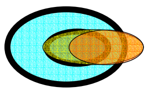
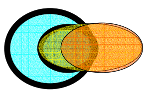
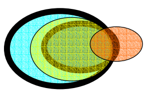
(17) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

	(64) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$		(65) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

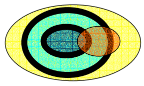
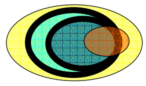
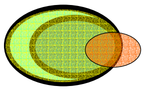
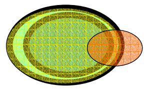
(18) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

	(66) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(67) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(68) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
	(69) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(70) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(71) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
	(72) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(73) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(74) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$
	(75) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(76) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(77) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

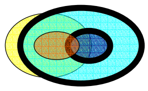
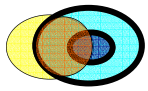
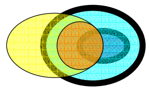
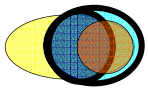
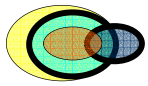
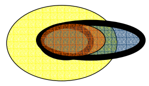
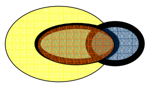
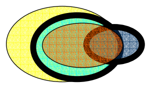
(19) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

	(78) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(79) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(80) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
	(81) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$		(82) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(83) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$
	(84) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		(85) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$		

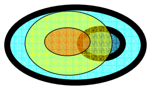
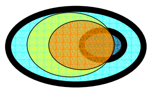
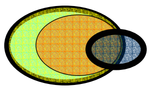
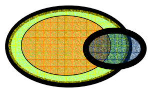
(20) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

	(86) $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$		(87) $\begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&1&0\\1&1&1&1\end{bmatrix}$
	(88) $\begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&1&0\\1&1&0&1\end{bmatrix}$		(89) $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\\1&1&0&1\end{bmatrix}$

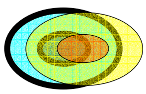
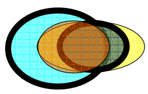
(21) $\begin{bmatrix}1&1&1\\1&1&1\\0&1&1\end{bmatrix}$

	(90) $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$		(91) $\begin{bmatrix}1&1&0&1\\1&1&0&1\\1&1&0&1\\0&1&1&1\end{bmatrix}$		(92) $\begin{bmatrix}1&1&0&1\\1&1&1&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$
	(93) $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&0&1\\0&1&1&1\end{bmatrix}$		(94) $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&0\\0&1&1&1\end{bmatrix}$		(95) $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&0&0&0\\0&1&1&1\end{bmatrix}$
	(96) $\begin{bmatrix}1&1&0&1\\1&1&0&1\\1&0&0&0\\0&1&1&1\end{bmatrix}$		(97) $\begin{bmatrix}1&1&0&1\\1&1&1&1\\1&1&1&0\\0&1&1&1\end{bmatrix}$		

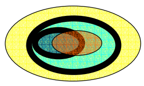
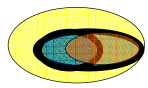
(22) $\begin{bmatrix}1&1&1\\1&1&1\\0&0&1\end{bmatrix}$

	(98) $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$		(99) $\begin{bmatrix}1&1&0&1\\1&1&1&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$
	(100) $\begin{bmatrix}1&1&0&1\\1&1&0&1\\1&1&1&0\\0&0&0&1\end{bmatrix}$		(101) $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&0\\0&0&0&1\end{bmatrix}$

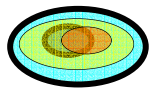
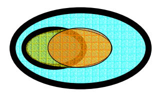
(23) $\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}$

	(102) $\begin{bmatrix}1&1&1&0\\1&1&1&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$		(103) $\begin{bmatrix}1&1&1&0\\1&1&1&1\\1&1&0&1\\0&1&1&1\end{bmatrix}$

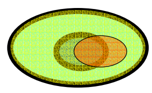
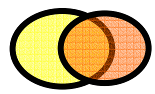
(24) $\begin{bmatrix}1&1&0\\1&1&0\\0&1&1\end{bmatrix}$

	(104) $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\\0&1&1&1\end{bmatrix}$		(105) $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&0&0&0\\0&1&1&1\end{bmatrix}$

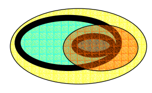
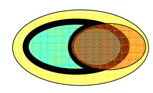
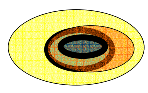
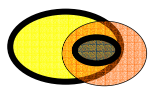
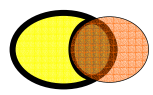
(25) $\begin{bmatrix}1&1&0\\1&1&1\\0&0&1\end{bmatrix}$

	(106) $\begin{bmatrix}1&1&1&0\\1&1&1&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$		(107) $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&1&0&1\\0&0&0&1\end{bmatrix}$

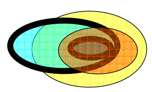
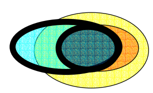
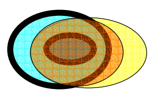
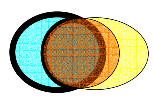
(26) $\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$

	(108) $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\\0&0&0&1\end{bmatrix}$		(109) $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$

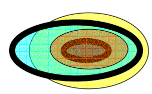
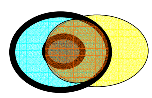
(27) $\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$

	(110) $\begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$		(111) $\begin{bmatrix}1&0&0&0\\1&1&1&0\\0&0&1&0\\1&1&1&1\end{bmatrix}$		(112) $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&0&0\\1&1&1&1\end{bmatrix}$
	(113) $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&0&1\end{bmatrix}$		(114) $\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\1&1&0&1\end{bmatrix}$		

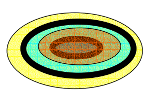
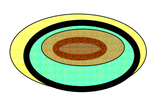
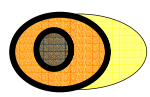
(28) $\begin{bmatrix}1&0&0\\1&1&1\\1&1&1\end{bmatrix}$

	(115) $\begin{bmatrix}1&0&0&0\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$		(116) $\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&1&1\\1&1&1&1\end{bmatrix}$
	(117) $\begin{bmatrix}1&0&0&0\\1&1&0&1\\1&1&0&1\\1&1&1&1\end{bmatrix}$		(118) $\begin{bmatrix}1&0&0&0\\1&1&0&1\\0&0&0&1\\1&1&1&1\end{bmatrix}$

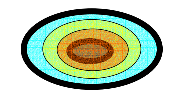
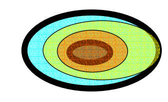
(29) $\begin{bmatrix}1&0&0\\1&1&1\\0&1&1\end{bmatrix}$

	(119) $\begin{bmatrix}1&0&0&0\\1&1&1&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$		(120) $\begin{bmatrix}1&0&0&0\\1&1&0&1\\1&1&0&1\\0&1&1&1\end{bmatrix}$

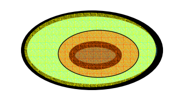
(30) $\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$

	(121) $\begin{bmatrix}1&0&0&0\\1&0&1&0\\1&1&1&0\\0&1&1&1\end{bmatrix}$		(122) $\begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&0\\0&1&1&1\end{bmatrix}$		(123) $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&0&0\\0&1&1&1\end{bmatrix}$

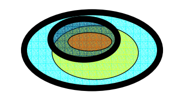
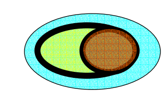
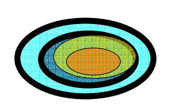
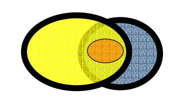
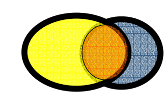
(31) $\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix}$

	(124) $\begin{bmatrix}1&0&0&0\\1&0&0&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$		(125) $\begin{bmatrix}1&0&0&0\\1&1&0&1\\1&1&1&1\\0&0&0&1\end{bmatrix}$

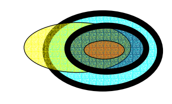
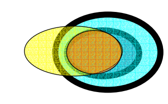
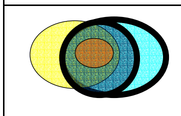
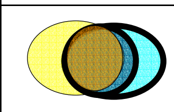
(32) $\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$

	(126) $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&0&0&1\end{bmatrix}$

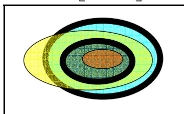
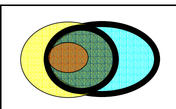
(33) $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

	(127) $\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&1&1&1\\0&0&0&1\end{bmatrix}$		(128) $\begin{bmatrix}1&1&0&1\\0&1&0&1\\0&1&1&1\\0&0&0&1\end{bmatrix}$		(129) $\begin{bmatrix}1&1&1&1\\0&1&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$
	(130) $\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{bmatrix}$		(131) $\begin{bmatrix}1&1&0&1\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}$		

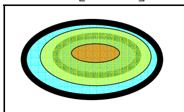
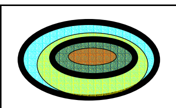
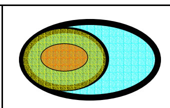
(34) $\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$

	(132) $\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{bmatrix}$		(133) $\begin{bmatrix}1&1&0&1\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{bmatrix}$
	(134) $\begin{bmatrix}1&1&0&1\\0&1&1&1\\0&0&0&1\\0&1&1&1\end{bmatrix}$		(135) $\begin{bmatrix}1&1&0&1\\0&1&0&1\\0&0&0&1\\0&1&1&1\end{bmatrix}$

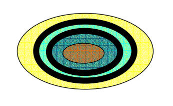
(35) $\begin{bmatrix}1&1&0\\0&1&1\\0&1&1\end{bmatrix}$

	(136) $\begin{bmatrix}1&1&1&0\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{bmatrix}$		(137) $\begin{bmatrix}1&1&1&0\\0&1&1&1\\0&0&0&1\\0&1&1&1\end{bmatrix}$

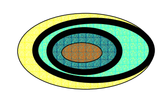
(36) $\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$

	(138) $\begin{bmatrix}1&1&1&0\\0&0&1&1\\0&1&1&1\\0&0&0&1\end{bmatrix}$		(139) $\begin{bmatrix}1&1&1&0\\0&1&1&1\\0&1&1&1\\0&0&0&1\end{bmatrix}$		(140) $\begin{bmatrix}1&1&1&0\\0&1&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$

(37) $\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix}$

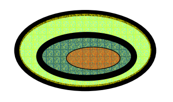


(141) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

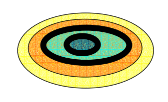


(142) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

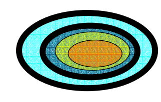
(38) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (39) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (40) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$



(143) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

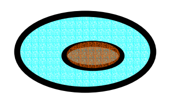


(144) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$



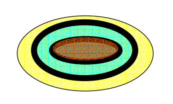
(145) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(41) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

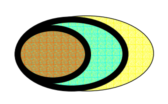


(146) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(42) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

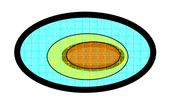


(147) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

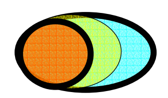


(148) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

(43) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

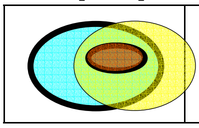


(149) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

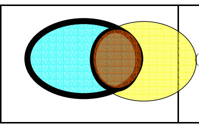


(150) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(44) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$



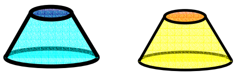
(151) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$



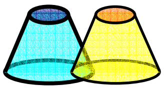
(152) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

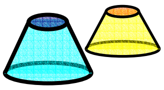
Appendix 3. Seventy-seven (77) topological relations between two real simple fuzzy regions in fuzzy topological space

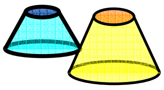
1.

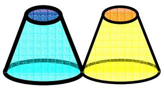
$(1)\begin{bmatrix} 0 & 0 & = \\ 0 & 0 & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

2.

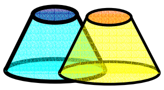
$(2)\begin{bmatrix} 0 & 0 & = \\ 0 & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

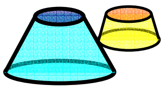
$(3)\begin{bmatrix} 0 & 0 & = \\ 0 & \subseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

$(4)\begin{bmatrix} 0 & 0 & = \\ 0 & \supseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

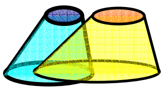
$(5)\begin{bmatrix} 0 & 0 & = \\ 0 & = & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

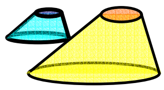
3.

$(6)\begin{bmatrix} 0 & \supseteq & = \\ 0 & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

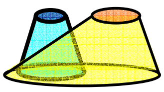
$(7)\begin{bmatrix} 0 & \supseteq & = \\ 0 & \subseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

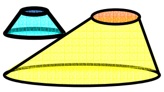
4.

$(8)\begin{bmatrix} 0 & \supseteq & 0 \\ 0 & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

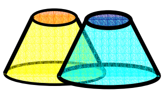
$(9)\begin{bmatrix} 0 & \supseteq & 0 \\ 0 & \supseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

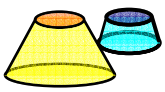
5.

$(10)\begin{bmatrix} 0 & \supseteq & 0 \\ 0 & \neq & 0 \\ = & \supseteq & = \end{bmatrix}$ 

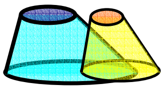
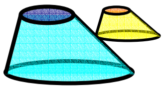
$(11)\begin{bmatrix} 0 & \supseteq & 0 \\ 0 & \supseteq & 0 \\ = & \supseteq & = \end{bmatrix}$ 

6.

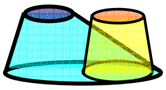
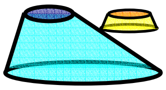
$(12)\begin{bmatrix} 0 & 0 & = \\ \subseteq & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

$(13)\begin{bmatrix} 0 & 0 & = \\ \subseteq & \subseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 

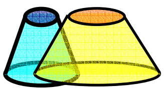
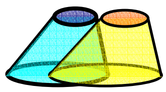
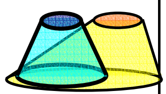
7.

(14) $\begin{bmatrix} 0 & 0 & = \\ \subseteq & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 	(15) $\begin{bmatrix} 0 & 0 & = \\ \subseteq & \subseteq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 

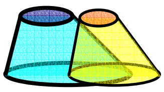
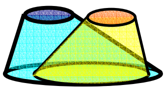
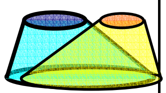
8.

(16) $\begin{bmatrix} 0 & 0 & = \\ \subseteq & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 	(17) $\begin{bmatrix} 0 & 0 & = \\ \subseteq & \subseteq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 

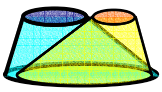
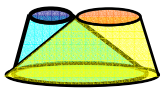
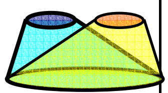
9. 10. 11.

(18) $\begin{bmatrix} 0 & \supseteq & = \\ \subseteq & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 	(19) $\begin{bmatrix} 0 & \supseteq & 0 \\ \subseteq & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 	(20) $\begin{bmatrix} 0 & \supseteq & 0 \\ \subseteq & \neq & 0 \\ = & \supseteq & = \end{bmatrix}$ 

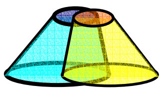
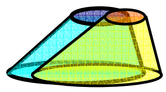
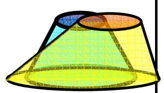
12. 13. 14.

(21) $\begin{bmatrix} 0 & \supseteq & = \\ \subseteq & \neq & 0 \\ = & \supseteq & = \end{bmatrix}$ 	(22) $\begin{bmatrix} 0 & \supseteq & = \\ \subseteq & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 	(23) $\begin{bmatrix} 0 & \supseteq & 0 \\ \subseteq & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 

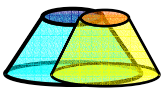
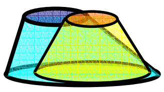
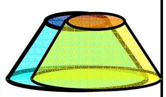
15. 16. 17.

(24) $\begin{bmatrix} 0 & \supseteq & 0 \\ \subseteq & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 	(25) $\begin{bmatrix} 0 & \supseteq & 0 \\ \subseteq & \neq & 0 \\ 0 & \supseteq & = \end{bmatrix}$ 	(26) $\begin{bmatrix} 0 & \supseteq & 0 \\ \subseteq & \neq & 0 \\ 0 & 0 & = \end{bmatrix}$ 

18. 19. 20.

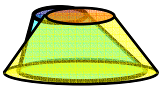
(27) $\begin{bmatrix} = & \supseteq & = \\ \subseteq & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 	(28) $\begin{bmatrix} = & \supseteq & 0 \\ \subseteq & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$ 	(29) $\begin{bmatrix} = & \supseteq & 0 \\ \subseteq & \neq & 0 \\ = & \supseteq & = \end{bmatrix}$ 

21. 22. 23.

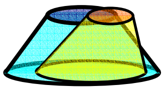
(30) $\begin{bmatrix} = & \supseteq & = \\ \subseteq & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 	(31) $\begin{bmatrix} = & \supseteq & = \\ \subseteq & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 	(32) $\begin{bmatrix} = & \supseteq & 0 \\ \subseteq & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 

24. 25. 26.

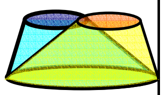
$$(33)\begin{bmatrix} = & \supseteq & 0 \\ \subseteq & \neq & 0 \\ 0 & \supseteq & = \end{bmatrix}$$



$$(34)\begin{bmatrix} = & \supseteq & 0 \\ \subseteq & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

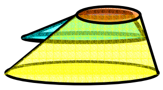


$$(35)\begin{bmatrix} = & \supseteq & 0 \\ \subseteq & \neq & 0 \\ 0 & 0 & = \end{bmatrix}$$

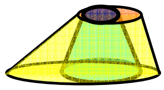


27.

$$(36)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \neq & 0 \\ = & \supseteq & = \end{bmatrix}$$

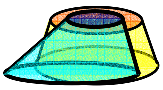


$$(37)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \subseteq & 0 \\ = & \supseteq & = \end{bmatrix}$$

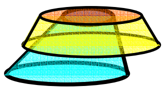


28.

$$(38)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \neq & \subseteq \\ = & \supseteq & = \end{bmatrix}$$

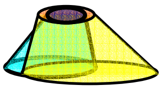


$$(39)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \subseteq & \subseteq \\ = & \supseteq & = \end{bmatrix}$$

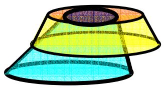


29.

$$(40)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$$

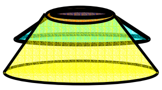


$$(41)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \subseteq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$$

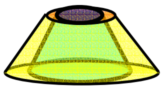


30.

$$(42)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \neq & 0 \\ 0 & \supseteq & = \end{bmatrix}$$

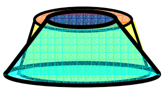


$$(43)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \subseteq & 0 \\ 0 & \supseteq & = \end{bmatrix}$$

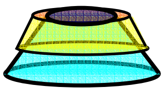


31.

$$(44)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

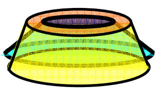


$$(45)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \subseteq & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

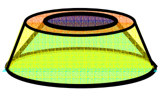


32.

$$(46)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \neq & 0 \\ 0 & 0 & = \end{bmatrix}$$

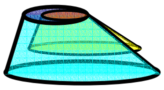


$$(47)\begin{bmatrix} = & 0 & 0 \\ \subseteq & \subseteq & 0 \\ 0 & 0 & = \end{bmatrix}$$

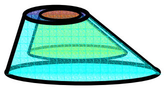


33.

$$(48)\begin{bmatrix} = & \supseteq & = \\ 0 & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

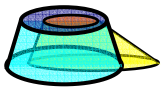


$$(49)\begin{bmatrix} = & \supseteq & = \\ 0 & \supseteq & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

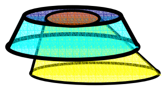


34.

$$(50)\begin{bmatrix} = & \supseteq & = \\ 0 & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$$

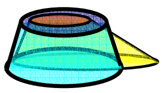


$$(51)\begin{bmatrix} = & \supseteq & = \\ 0 & \supseteq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$$

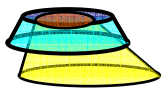


35.

$$(52)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$$

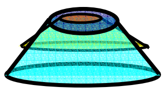


$$(53)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \supseteq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$$

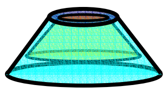


36.

$$(54)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

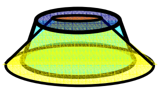


$$(55)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \supseteq & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

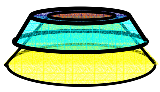


37.

$$(56)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \neq & 0 \\ 0 & \supseteq & = \end{bmatrix}$$

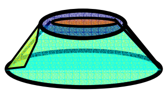


$$(57)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \supseteq & 0 \\ 0 & \supseteq & = \end{bmatrix}$$

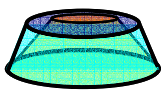


38.

$$(58)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \neq & 0 \\ 0 & 0 & = \end{bmatrix}$$

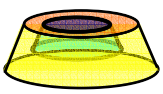


$$(59)\begin{bmatrix} = & \supseteq & 0 \\ 0 & \supseteq & 0 \\ 0 & 0 & = \end{bmatrix}$$

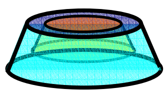


39. 40

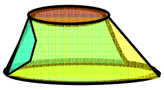
$$(60)\begin{bmatrix} 0 & 0 & = \\ 0 & 0 & \subseteq \\ = & \supseteq & = \end{bmatrix}$$

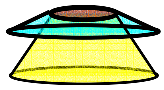


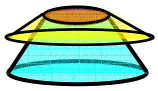
$$(61)\begin{bmatrix} = & \supseteq & = \\ 0 & 0 & \subseteq \\ 0 & 0 & = \end{bmatrix}$$

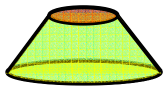


41.

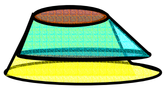
(62) $\begin{bmatrix} = & 0 & 0 \\ 0 & \neq & 0 \\ 0 & 0 & = \end{bmatrix}$ 

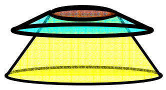
(63) $\begin{bmatrix} = & 0 & 0 \\ 0 & \supseteq & 0 \\ 0 & 0 & = \end{bmatrix}$ 

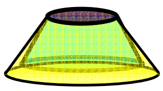
(64) $\begin{bmatrix} = & 0 & 0 \\ 0 & \subseteq & 0 \\ 0 & 0 & = \end{bmatrix}$ 

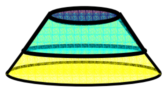
(65) $\begin{bmatrix} = & 0 & 0 \\ 0 & = & 0 \\ 0 & 0 & = \end{bmatrix}$ 

42.

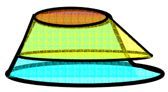
(66) $\begin{bmatrix} = & 0 & 0 \\ 0 & \neq & 0 \\ 0 & \supseteq & = \end{bmatrix}$ 

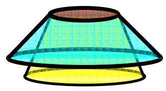
(67) $\begin{bmatrix} = & 0 & 0 \\ 0 & \supseteq & 0 \\ 0 & \supseteq & = \end{bmatrix}$ 

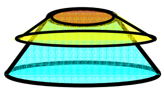
(68) $\begin{bmatrix} = & 0 & 0 \\ 0 & \subseteq & 0 \\ 0 & \supseteq & = \end{bmatrix}$ 

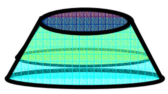
(69) $\begin{bmatrix} = & 0 & 0 \\ 0 & = & 0 \\ 0 & \supseteq & = \end{bmatrix}$ 

43.

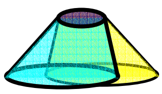
(70) $\begin{bmatrix} = & 0 & 0 \\ 0 & \neq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 

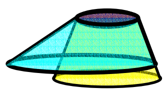
(71) $\begin{bmatrix} = & 0 & 0 \\ 0 & \supseteq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 

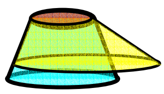
(72) $\begin{bmatrix} = & 0 & 0 \\ 0 & \subseteq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 

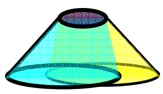
(73) $\begin{bmatrix} = & 0 & 0 \\ 0 & \subseteq & \subseteq \\ 0 & 0 & = \end{bmatrix}$ 

44.

(74) $\begin{bmatrix} = & 0 & 0 \\ 0 & \neq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 

(75) $\begin{bmatrix} = & 0 & 0 \\ 0 & \supseteq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 

(76) $\begin{bmatrix} = & 0 & 0 \\ 0 & \subseteq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 

(77) $\begin{bmatrix} = & 0 & 0 \\ 0 & \subseteq & \subseteq \\ 0 & \supseteq & = \end{bmatrix}$ 

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Samenvatting

Op dit moment worden natuurlijke fenomenen door geografische informatie systemen gerepresenteerd als scherpe ruimtelijke objecten. In feite hebben veel natuurlijke fenomenen echter vage eigenschappen. Hun representatie in een scherpe vorm versimpelt aanzienlijk de manier waarop ze kunnen worden verwerkt. Deze vereenvoudiging kan de natuurlijke karakteristieken echter onvoldoende waarborgen en leidt tot verlies van informatie. Om natuurlijke fenomenen meer nauwkeurig te beschrijven zou de vaagheid van de objecten in beschouwing moeten worden betrokken en worden gepresenteerd door het GIS. Dit moet dan leiden tot betere resultaten en een beter begrip van de wereld om ons heen.

Het centrale thema van dit proefschrift is het onderbrengen van vage ruimtelijke objecten in vanuit GIS. Verschillende onderwerpen worden bediscussieerd vanuit een theoretisch en een praktisch perspectief. Vage ruimtelijke objecten, alsmede de topologische relaties ertussen, worden gedefinieerd. Ze worden gemodelleerd, gegenereerd en gebruikt voor het modelleren van verandering in land bedekking.

Een formele definitie van scherpe ruimtelijke objecten wordt afgeleid op basis van abstracte wiskunde, zoals verzamelingleer en topologie. Vage verzamelingleer en vage topologie zijn perfecte hulpmiddelen om vage ruimtelijke objecten theoretisch te definiëren, omdat vage verzamelingleer een natuurlijke uitbreiding is van de klassieke verzamelingleer en vage topologie gebaseerd is op vage verzamelingen. Met betrekking tot deze uitbreiding moeten verschillende eigenschappen die gelden voor scherpe verzamelingen worden heroverwogen voor vage verzamelingen.

Het belangrijkste onderdeel van een vaag ruimtelijk object is zijn grens. Drie definities van vage grenzen worden opnieuw bekeken en één ervan is hierbij geselecteerd voor het definiëren van vage ruimtelijke objecten. Behalve de vage grens komen verschillende begrippen van vage ruimtelijke objecten aan de orde, zoals de kern, het inwendige, het randgebied, het inwendige van het randgebied en het complement van een vage verzameling in een vage topologische ruimte. De relaties tussen deze begrippen en het inwendige, de grens en het complement van een vage verzameling worden ten tonele gevoerd. De kern kan gezien worden als de scherpe deelverzameling van een vage verzameling en het grensgebied is een soort grens, die echter een fijnere structuur laat zien dan de grens van een vage verzameling in een vage topologische ruimte. Van al deze concepten wordt bewezen dat het topologische eigenschappen zijn van een vage topologische ruimte.

Er wordt aangetoond dat de definitie van een enkelvoudig vaag gebied gebaseerd is op de bovengenoemde topologische eigenschappen. In dit proefschrift wordt het twee keer besproken. Allereerst wordt de definitie van een vaag gebied gegeven in een speciale topologische ruimte, de zogenaamde scherp-vage topologische ruimte. Deze heet zo omdat de meeste toplogische eigenschappen van vage verzamelingen dezelfde zijn als die we tegenkomen in een scherpe topologische ruimte. Een formele definitie van een enkelvoudig vaag gebied wordt voorgesteld op basis van de bespreking van topologische eigenschappen, naast het inwendige, de grens en het complement van een vage verzameling in de algemene toplogische ruimte. Een enkelvoudig scherp gebied is een speciaal geval van een enkelvoudig vaag gebied.

Fundamentele eigenschappen met betrekking tot vage ruimtelijke objecten betreffen de topologische relaties. Het probleem van de benadering via 9-doorsneden voor het identificeren van topologische relaties tussen vage ruimtelijke objecten wordt besproken. Om topologische relaties tussen vage ruimtelijke objecten af te leiden wordt deze benadering via 9-doorsneden verfijnd tot een benadering met 3*3-doorsneden in de scherpe topologische ruimte. Vervolgens wordt een benadering met een matrix van 4*4-doorsneden opgebouwd, waarbij gebruik gemaakt wordt van de topologische eigenschappen van vage verzamelingen. Een matrix van 5*5-doorsneden kan vervolgens worden geconstrueerd op basis van een bepaalde voorwaarde in de scherp-vage topologische ruimte. Deze matrices worden dan aangepast binnen de algemene vage topologische ruimte, gebaseerd op topologische eigenschappen die verschillen van het inwendige, de grens en het complement van 2 vage verzamelingen. Twee matrices van 3*3-doorsneden en één matrix van 4*4-doorsneden worden geïntroduceerd in de algemene topologische ruimte. De topologische relaties tussen enkelvoudige vage gebieden kunnen vervolgens worden geïdentificeerd op basis van topologische invariantie in de doorsneden van doorsnijdingsmatrices. Door gebruik te maken van lege en niet-lege topologische invariantie van de doorsnijdingsmatrices worden respectievelijk 44 en 152 relaties afgeleid tussen twee enkelvoudige vage gebieden.

Het modelleren van vage ruimtelijke objecten moet niet alleen gedaan voor enkelvoudige vage gebieden, maar ook voor vage lijnen en vage punten. Een vage cel wordt geïntroduceerd om vage lijnen en vage punten te modelleren alsmede de relaties tussen vage ruimtelijke objecten. Een complex van vage cellen kan dan worden opgebouwd uit vage cellen. Een vaag gebied, een vage lijn en een vaag punt worden dan gedefinieerd op basis van deze structuur. De relaties tussen vage objecten worden vastgesteld. De structuur van een complex van vage cellen biedt een theoretisch kader omdat daarmee vage ruimtelijke objecten gemakkelijk kunnen worden gemodelleerd.

Na het vaststellen van een theoretisch kader voor het modelleren van vage ruimtelijke objecten worden in dit proefschrift verschillende praktische aspecten besproken die relevant zijn bij het toepassen van vage ruimtelijke objecten. Het eerste probleem is hoe vage ruimtelijke objecten gegenereerd kunnen worden. Een samengestelde methode wordt geïntroduceerd om vage objecten voor land bedekking te genereren. Het bestaat uit verschillende fasen, vanaf het ontwerpen van lidmaatschapfuncties tot de classificatie en het verfijnen van lidmaatschapfuncties voor vage objecten voor land bedekking.

Een ander praktisch probleem betreft het terugvinden van vage ruimtelijke objecten, vooral als dat moet gebeuren op basis van topologische relaties. In een traditioneel GIS worden de zoekoperaties gedefinieerd op basis van een relatief klein aantal topologische relaties. Er bestaan echter vele topologische relaties tussen vage ruimtelijke objecten. Om vage ruimtelijke gebieden op te vragen worden zoekoperatoren voorgesteld en geformaliseerd op basis van gebruikelijke operatoren in een traditioneel GIS. De 44 of

152 topologische relaties worden door middel van 4 verschillende methoden onder deze operatoren geclassificeerd. Deze methoden geven een betrekkelijk volledige verzameling voor het bevragen van vage ruimtelijke objecten om tegemoet te komen aan de vereisten van de verschillende toepassingen.

Het derde praktische probleem betreft het gebruik van vage ruimtelijke objecten in realistische toepassingen. De dynamiek van land bedekkingen is een erg belangrijk onderwerp in China. Het accent ligt hierbij op het verkennen van de veranderingen. De stad Sanya, in het zuiden van China, is uitgezocht als een testgebied. Een vage redeneringmethode wordt voorgesteld om veranderingen in land bedekking te berekenen. Er wordt aangetoond dat met een vage representatie niet alleen betere resultaten worden geboekt voor het berekenen van de veranderingen in land bedekking maar ook de details van de veranderingen worden belicht.

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